Revenue Monotonicity in Core-Selecting Package Auctions

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Abstract

Both the Vickrey auction and minimum-revenue core-selecting auctions suffer from problems of revenue monotonicity: adding bidders or increasing bids can decrease the seller's revenue. This paper gives necessary and sufficient conditions for monotonicity in these mechanisms and then proposes multiple monotonic core-selecting auctions. One such class of auctions, called "maximin-revenue" auctions, is shown to provide optimal bidder incentives within the class of all monotonic, core-selecting mechanisms.
Key words: Vickrey mechanism, core-selecting package auctions, revenue monotonicity

1. Introduction

Package auctions allow bidders to place bids on groups of the items for sale, rather than on individual items only. This feature allows bidders to express preferences that include complements or substitutes and shields them from the risks that arise when an auction ignores these kinds of valuations. Bidding individually on two goods that are substitutes creates a risk that the bidder will win both goods, while bidding individually on two goods that are complements creates a risk that the bidder will only win one of the two. These risks are not the only problems that arise with valuations that are not additively separable in the goods. Complementarities produce situations where

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market-clearing prices do not exist,[1] so that standard dynamic auctions do not have well-defined stopping points. A package auction corrects all of these complications.

The famous Vickrey auction is one such package auction. In fact, it is the only direct mechanism for which truthful reporting is a dominant strategy, the allocation is efficient, and losing bidders do not receive payments.[2] Despite the fact that the Vickrey mechanism so successfully implements truthful reporting in dominant strategies, it is rarely used in practice due to its vulnerability to low and non-monotonic seller revenue, shill bidding, and collusion.[3] Both low revenue and shill bidding are related to the fact that the Vickrey outcome does not always lie within the core. There is some coalition of bidders that would prefer a different allocation scheme and would pay the seller more than his Vickrey revenue for it.

Day and Milgrom (2008) proposed core-selecting package auctions as alternatives to the Vickrey auction that correct some of these weaknesses. Core-selecting package auctions choose allocations and payments so that payoffs lie in the core. In particular, Day and Milgrom (2008) advocate *minimum-revenue core-selecting auctions* that give the seller his smallest possible revenue in the core. The minimum-revenue core-selecting auctions alleviate the low revenue problem but, due to their close connection to the Vickrey auction, still fall victim to non-monotonic revenue: adding bidders or increasing reported values can decrease the seller's revenue. Complement valuations are not the only causes of non-monotonic revenue. As we show in the example in Section 3.1, the Vickrey auction is not monotonic in reported values even when restricted to substitutable valuations, in which case the Vickrey and minimum core revenue outcomes coincide.

Monotonicity of the seller's revenue in bidders and reported values is important because it reduces incentives for the seller and buyers to try to game the mechanism. When revenue is not monotonic, sellers may want to disqualify bids or bidders to

[1]See Gul and Stacchetti (1999) and Milgrom (2000).

[2]See Green and Laffont (1979) and Holmstrom (1979).

[3]For a detailed treatment of these weaknesses, see Ausubel and Milgrom (2006).

increase prices while bidders may want to raise their bids or introduce shills to reduce prices. Any of these actions can destroy the efficiency of an auction's outcome.

Moreover, with direct package mechanisms, the bidders' abilities to trust the seller plays an important role because items are awarded and prices are calculated in a "black box" procedure run by the seller. Potential bidders may decide not to participate in auctions that give the seller large incentives to cheat.

The existing literature on revenue monotonicity has studied the effect of adding bidders to the auction on the seller's revenue. We call this *bidder revenue monotonicity*. Papers have reported necessary and sufficient conditions for bidder revenue monotonicity in the Vickrey auction, minimum-revenue core-selecting auctions, and dominant-strategy mechanisms. Ausubel and Milgrom (2006) proved that if the set of possible valuation profiles includes all additive valuations, then the Vickrey outcome is bidder revenue monotonic if and only if all bidders have substitutes valuations. Lamy (2010) demonstrated that having only two goods for sale implies that minimum-revenue core-selecting auctions exhibit bidder revenue monotonicity. We discuss these conditions – along with a new sufficient condition for bidder revenue monotonicity in minimum-revenue auctions – in Section 3.

Other authors have given up efficiency to study bidder revenue monotonicity in dominant-strategy auction mechanisms. Rastegari et al. (2007, 2009a) give a quasi-impossibility result. Under certain conditions, no dominant-strategy mechanism is bidder revenue monotonic. Todo et al. (2009a,b) discover a restriction on the allocation rule that is both necessary and sufficient for dominant-strategy mechanisms to exhibit bidder revenue monotonicity. Taking the analysis a step further, Rastegari et al. (2009b) suggest a randomized, dominant-strategy mechanism that achieves bidder revenue monotonicity in expectation.

This paper forgoes incentive-compatibility in order to focus on core-selecting mechanisms, which preserve efficiency and the possibility of monotonicity. More importantly, this paper is interested not only in bidder revenue monotonicity but also in changes in the seller's revenue due to changes in the reported values. We call this *revenue* *monotonicity* and believe it is as important as bidder revenue monotonicity because disqualifying bids creates the same problems as disqualifying bidders. Consequently, we provide a new necessary condition for the Vickrey auction to exhibit revenue monotonicity and show that it essentially restricts us to additively separable valuations. We turn back to the core and propose multiple revenue- and bidder-revenue-monotonic core-selecting mechanisms. We focus on one class of such mechanisms, which we call maximin revenue, because it minimizes marginal and maximum (non-marginal) incentives to deviate from truth-telling. The derivation of these maximin-revenue core-selecting auctions indicates that we do not need to fully characterize the core to find a bidder-optimal revenue-monotonic mechanism. We can create such an auction mechanism using Vickrey payments over a certain set of valuation profiles.

The remainder of this paper is organized as follows. Section 2 sets up the package auction problem, defines the core, and describes the class of minimum-revenue core-selecting auctions. Section 3 defines our two monotonicity concepts and provides necessary and sufficient conditions for monotonicity of the Vickrey auction and minimum-revenue core-selecting auctions. Section 4 presents several monotonic core-selecting mechanisms. In particular, it discusses maximin-revenue auctions, detailing how to calculate the maximin revenue and characterizing bidder incentives in these auctions. Section 5 concludes.

2. The Package Auction Model and Core-Selecting Mechanisms

We consider an auction setting with a seller, agent 0, and a set of bidders $N = \{1, \ldots, n\}$. The set of goods being auctioned is $K = \{1, \ldots, k\}$. The set of feasible packages available for bidder $i \in N$ to win is denoted X_i . We might have $X_i = 2^K$ so that bidder i can win any subset of the available goods. This notation also allows for more restrictions on the set of packages available to each bidder and, therefore, we will focus not on the set of goods being auctioned but on the set of feasible allocations

$$X = \{(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n | x_i \cap x_j = \emptyset \ \forall i \neq j\} .$$

For all $i \in N$, we require $\emptyset \in X_i$ so that X is always non-empty.

Both bidders and the seller have quasi-linear utility. Bidders' values for available packages are determined by a valuation profile.

Definition 1. *A valuation profile is a function* $u = (u_1, \ldots, u_n)$ *such that for all* $i \in N, u_i : X_i \to \mathbb{R}_+, u_i(\emptyset) = 0$*, and if* $x, y \in X_i$ *with* $x \subseteq y$*, then* $u_i(x) \leq u_i(y)$*.*

Valuation profiles must assign a weakly positive value to each possible package, are normalized so that winning nothing provides zero utility, and satisfy free disposal – more is always weakly better. Bidders' values are independent and they are private information. We assume the seller has no value for the goods being auctioned. Given a goods assignment $x = (x_1, \ldots, x_n) \in X$ and payments $p_i \in \mathbb{R}_+$ from each bidder to the seller, we denote the seller's payoff as $\pi_0 = \sum_{i=1}^{n} p_i$ and the bidders' payoffs as $\pi_i = u_i(x_i) - p_i$ for all $i \in N$.

For any set of bidders $S \subseteq N$, define the value of the coalition consisting of that set of bidders and the seller as

$$w_u(S) = \max_{x \in X} \sum_{i \in S} u_i(x_i). \tag{1}$$

Also define $w_u(\emptyset) = 0$. Notice that in this notation, S does not include the seller even though he is implicitly part of the value. This simplification works in the auction setting because no group of bidders S has any value without the seller, so the coalitional values $w_u(S)$ defined above are the only relevant values.

The *core* is the set of imputations that are feasible and not blocked by any coalition (i.e., each coalition receives at least as much as it could achieve on its own, which is either $w_u(S)$ or zero, depending on whether the seller is part of the coalition):

$$Core(N, u) = \left\{ \pi = (\pi_0, \pi_1, \ldots, \pi_n) \geq 0 : \sum_{i=0}^{n} \pi_i \leq w_u(N), \pi_0 + \sum_{i \in S} \pi_i \geq w_u(S) \ \forall S \subseteq N \right\}. \tag{2}$$

Solutions in the core are stable in the sense that no group of bidders and the seller would prefer to abandon the outcome of the auction in favor of some alternative deal among its members. Notice that every allocation corresponding to a core imputation is efficient:

$$\pi \in Core(N,u) \Rightarrow \sum_{i=1}^{n} u_i(x_i) = \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} (u_i(x_i) - p_i) = \sum_{i=0}^{n} \pi_i = w_u(N)$$
$$= \max_{\hat{x} \in X} \sum_{i \in N} u_i(\hat{x}_i) \, .$$

Also, the core always exists because, given an efficient allocation x^*, the imputation $\pi_0 = \sum_{i=1}^{n} u_i(x_i^*) = w_u(N)$ and $\pi_i = 0$ for all $i \in N$ is always in the core. This is the seller's preferred imputation. Bidders may not all agree on their preferred imputations. This motivates the following definition.

Definition 2. *An imputation $\pi \in Core(N,u)$ is* bidder optimal *if there is no $\hat{\pi} \in Core(N,u)$ such that $\hat{\pi}_i \geq \pi_i$ for all $i \in N$ and the inequality is strict for at least one bidder.*

A bidder prefers imputations in the core in which he receives his Vickrey payoff: $\pi_i^v(u) = w_u(N) - w_u(N \setminus \{i\})$. This payoff is achievable in the core because the imputation $\pi_0 = w_u(N \setminus \{i\}), \pi_i = w_u(N) - w_u(N \setminus \{i\}), \pi_j = 0$ for all $j \neq i$, always lies in the core. If $i \in S$, then $\pi_0 + \sum_{i \in S} \pi_i = w_u(N) \geq w_u(S)$ and if $i \notin S$, then $\pi_0 + \sum_{i \in S} \pi_i = w_u(N \setminus \{i\}) \geq w_u(S)$. No bidder can ever receive more than his Vickrey payoff in any core imputation. If $\pi_i > w_u(N) - w_u(N \setminus \{i\})$, then

$$\pi_0 + \sum_{j \neq i} \pi_j = w_u(N) - \pi_i < w_u(N) - w_u(N) + w_u(N \setminus \{i\}) = w_u(N \setminus \{i\}) \, ,$$

a contradiction to π being in the core. Thus, when the Vickrey imputation

$$\pi^v(u) = (\pi_0^v(u), \ldots, \pi_n^v(u))$$
$$= \left(w_u(N) - \sum_{i=1}^{n} \left[w_u(N) - w_u(N \setminus \{i\}) \right], w_u(N) - w_u(N \setminus \{1\}), \ldots, w_u(N) - w_u(N \setminus \{n\}) \right)$$

lies in the core, it is the unique bidder-optimal imputation. When it is not in the core, there are multiple bidder optimal imputations.

This paper considers package auctions that select outcomes in the core. A *core-selecting auction* is a direct mechanism $(x(\hat{u}), p(\hat{u})) \in (X, \mathbb{R}^n_+)$ that selects an allocation and bidder payments such that the imputation $\pi_0 = \sum_{i=1}^n p_i(\hat{u}), \pi_i = \hat{u}_i(x_i(\hat{u})) - p_i(\hat{u})$ is in the core with respect to the reported values $\hat{u}$. One class of core-selecting auctions includes those that minimize the seller's revenue.

Definition 3. *A* minimum-revenue core-selecting auction *is a core-selecting auction that chooses* $(x(\hat{u}), p(\hat{u})) \in (X, \mathbb{R}^n_+)$ *such that the seller receives the minimum possible revenue within the set of core imputations:*

$$\pi_0 = \sum_{i=1}^n p_i(\hat{u}) = \min \{\pi_0 | \pi \in Core(N, \hat{u})\} .$$

Finding the seller's minimum revenue at any core allocation is equivalent to solving the following problem:

$$\min_{\pi \geq 0} \{\pi_0\} \text{ subject to } \sum_{i=0}^n \pi_i = w_u(N) \text{ and } \pi_0 + \sum_{i \in S} \pi_i \geq w_u(S) \quad \forall S \subseteq N \tag{3}$$

Using the equality constraint to replace π_0 with $w_u(N) - \sum_{i=1}^n \pi_i$, problem (3) becomes:

$$\min_{\pi \geq 0} \left\{ w_u(N) - \sum_{i=1}^n \pi_i \right\} \text{ subject to } w_u(N) - w_u(N \setminus \{S\}) \geq \sum_{i \in S} \pi_i \quad \forall S \subseteq N ,$$

or equivalently,

$$\max_{\pi \geq 0} \left\{ \sum_{i=1}^n \pi_i \right\} \text{ subject to } w_u(N) - w_u(N \setminus \{S\}) \geq \sum_{i \in S} \pi_i \quad \forall S \subseteq N. \tag{4}$$

In what follows, this latter formulation will be the easiest way of thinking about the minimum-revenue problem.

Minimum-revenue core-selecting auctions always choose bidder-optimal imputations. Therefore, when the Vickrey imputation lies in the core, it is the unique outcome of any minimum-revenue core-selecting auction. This connection to the Vickrey mechanism causes minimum-revenue core-selecting auctions to face the same revenue monotonicity problems as the Vickrey mechanism.

3. Revenue Monotonicity

To define revenue monotonicity, we must first establish a partial ordering for valuation profiles.

Definition 4. *We say one valuation profile* $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$ *is lower than another* $u = (u_1, \ldots, u_n)$ *if* $\hat{u}_i(x_i) \leq u_i(x_i)$ *for all* $i \in N$ *and all possible bundles* $x_i \in X_i$. *We will denote this by* $\hat{u} \leq u$. *If for some* $i \in N$ *and some* $x_i \in X_i$, $\hat{u}_i(x_i) < u_i(x_i)$, *then we write* $\hat{u} < u$.

Now we are ready to define two monotonicity concepts, revenue monotonicity and bidder revenue monotonicity.

Definition 5. *A mechanism is* revenue monotonic *if* $\hat{u} \leq u$ *implies that* $\pi_0(\hat{u}) \leq \pi_0(u)$.

Definition 6. *A mechanism is* bidder revenue monotonic *if* $\hat{u} = u_{-j} = (u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n)$ *for any* $j \in N$ *implies that* $\pi_0(\hat{u}) \leq \pi_0(u)$.

Revenue monotonicity does not always imply bidder revenue monotonicity because the outcome of a mechanism may depend directly on the number of participating bidders. Notice that $\pi_0(u)$ is implicitly a function of the set of participating bidders N and we could have written it as $\pi_0(u, N)$. Let $\bar{N}$ be the set of all possible bidders in the world. Then for any set of participating bidders $N \subseteq \bar{N}$ and any valuation profile $u : X \to \mathbb{R}_+^{\bar{N}}$, let $u_i^N(x_i) = u_i(x_i)$ if $i \in N$ and $u_i^N(x_i) = 0$ if $i \notin N$. Revenue monotonicity implies bidder revenue monotonicity if $\pi_0(u, N) = \pi_0(u^N, \bar{N})$. In other words, the mechanism selects the same outcome when all non-participating bidders enter and report values of zero for all packages.

3.1. The Vickrey Mechanism

It is well known that the Vickrey mechanism is neither revenue monotonic nor bidder revenue monotonic. For example, suppose we have three bidders and two goods for sale. Bidder 1 wants only good 1 and has a value of 5 for it. Likewise, bidder 2 wants only good 2 and has a value of 5 for it. Bidder 3 considers goods 1 and 2 to be complements. He has a value of 5 for the package, but a value of zero if he receives only one item. The efficient allocation assigns good 1 to bidder 1 and good 2 to bidder 2. The Vickrey payments are $u_i(x_i^*) - (w_u(N) - w_u(N \setminus \{i\})) = 5 - (10 - 5) = 0$ for $i = 1, 2$, so the seller receives zero revenue. If bidder 1 does not participate or his value for good 1 is reduced to zero, both goods are assigned to bidder 3 and he pays

$$u_3(x_3^*) - (w_u(N) - w_u(N \setminus \{3\})) = 5 - (5 - 5) = 5.$$

The seller's revenue strictly increases.

This popular example shows only that the Vickrey mechanism is not revenue monotonic for $n \geq 3$. As it turns out, the Vickrey mechanism is also not revenue monotonic if we restrict to two bidders. The following example with two bidders, 1 and 2, and two goods, A and B, demonstrates the opportunity for non-monotonicities.

	A	B	AB	p_i
Bidder 1	**4**	2	4	1
Bidder 2	2	**2**	3	0

$\rightarrow$

	A	B	AB	p_i
Bidder 1	**3**	2	4	1
Bidder 2	2	**2**	3	1

The efficient allocations are shown in bold. When bidder 1's value for good A decreases from 4 to 3, the seller's revenue increases from 1 to 2.

Ausubel and Milgrom (2006) provide a necessary and sufficient condition on valuation profiles for the Vickrey mechanism to be bidder revenue monotonic. To understand the condition, we must define additive and substitutable valuations:

Definition 7. *We say that a bidder's valuation is* additive *if for any packages $x_i, y_i \in X_i$ such that $x_i \cap y_i = \emptyset$, $u_i(x_i \cup y_i) = u_i(x_i) + u_i(y_i)$.*

Definition 8. *We say that a bidder's valuation is* substitutable *if for any packages $x_i, y_i \in X_i$, $u_i(x_i \cap y_i) + u_i(x_i \cup y_i) \leq u_i(x_i) + u_i(y_i)$.*

Theorem 1 (Ausubel and Milgrom (2006)). *Let U be the set of possible individual value functions and suppose U includes all additive valuations. Then all valuations* *in U are substitutable if and only if the Vickrey mechanism is bidder revenue monotonic on the domain U^N.*

Note that this condition does not guarantee revenue monotonicity, as demonstrated by the previous example with two bidders in which both valuations were substitutable. The following proposition gives a necessary condition for the seller's Vickrey payoff to exhibit revenue monotonicity. This condition says that bidder i's value for the package he wins in the efficient allocation adds weakly more to smaller sets of bidders (of size $n-1$) than to the grand coalition. To write this condition formally, we must define coalitional values that depend directly on the feasible set of allocations.

Definition 9.

- *For $S \subseteq N$ and $Y \subseteq X$, let $w_u(S,Y) = \max_{x \in Y} \sum_{i \in S} u_i(x_i)$.*
- *Also, for any package $x_i \in X_i$, let $Y \setminus \{x_i\} = \{y \in Y | x_i \nsubseteq y\}$.*

Proposition 1. *The following condition is necessary for the seller's Vickrey payoff to be revenue monotonic:*
for all $i \in N$, $j \neq i$, and $x^* \in \arg\max_{x \in X} \sum_{k \in N} u_k(x_k)$

$$w_u(N,X) - w_u(N, X \setminus \{x_i^*\}) \leq w_u(N \setminus \{j\}, X) - w_u(N \setminus \{j\}, X \setminus \{x_i^*\}). \qquad (5)$$

Proof: Suppose (5) does not hold. Consider the valuation profile

- $\hat{u} = (\hat{u}_i, u_{-i})$ such that $\hat{u}_i(x_i) = u_i(x_i)$ for all $x_i \nsupseteq x_i^*$ and
- $\hat{u}_i(x_i) = u_i(x_i) - (w_u(n,X) - w_u(N, X \setminus \{x_i^*\}))$ for all $x_i \supseteq x_i^*$.

By construction, $x^* \in \arg\max_{x \in X} \sum_{k \in N} \hat{u}_k(x_k)$ and $w_{\hat{u}}(N) = w_{\hat{u}}(N,X) = w_{\hat{u}}(N, X \setminus \{x_i^*\})$. Now

$$\pi_0^v(u) = w_u(N) - \sum_{k=1}^{n} (w_u(N) - w_u(N \setminus \{k\})) = \sum_{k=1}^{n} w_u(N \setminus \{k\}) - (n-1) w_u(N).$$

Decreasing the value of the grand coalition by $\Delta = w_u(N,X) - w_u(N, X \setminus \{x_i^*\})$ increases the second term of π_0^v, $-(n-1)w_u(N)$, by $(n-1)\Delta$. If the first term, $\sum_{k=1}^{n} w_u(N \setminus \{k\})$, decreases by less than $(n-1)\Delta$, then we get a net increase in π_0^v. By construction of $\hat{u}$, no $w_u(N \setminus \{k\})$ can decrease by more than Δ. Moreover, $\hat{w}_u(N \setminus \{i\}) = w_u(N \setminus \{i\})$ because only bidder i's values have changed. Then

$$
\begin{aligned}
\sum_{k=1}^{n} w_u(N \setminus \{k\}) - \sum_{k=1}^{n} \hat{w}_u(N \setminus \{k\}) &= \sum_{k \in N \setminus \{i\}} w_u(N \setminus \{k\}) - \sum_{k \in N \setminus \{i\}} \hat{w}_u(N \setminus \{k\}) \\
&\leq (n-2)\Delta + w_u(N \setminus \{j\}) - \hat{w}_u(N \setminus \{j\}) \\
&\leq (n-2)\Delta + w_u(N \setminus \{j\}) - \hat{w}_u(N \setminus \{j\}, X \setminus \{x_i^*\}) \\
&< (n-2)\Delta + w_u(N) - w_u(N, X \setminus \{x_i^*\}) \\
&= (n-1)\Delta\,.
\end{aligned}
$$

The second inequality follows from the fact that $\hat{w}_u(N \setminus \{j\}) \geq w_u(N \setminus \{j\}, X \setminus \{x_i^*\})$ because the values of all packages in $X \setminus \{x_i^*\}$ have not changed. The third inequality follows from (5) being violated. Thus, $\pi_0^v(\hat{u}) > \pi_0^v(u)$ even though $\hat{u} < u$, a contradiction. ■

Since decreasing the valuation profile weakly decreases all coalition values, the only way to increase $\pi_0^v(u) = \sum_{k=1}^{n} w_u(N \setminus \{k\}) - (n-1)w_u(N)$ is to decrease the value of the efficient allocation, thereby decreasing $w_u(N)$. The proof of Proposition 1 establishes that the only way a reduction of $w_u(N)$ leads to additional seller revenue is if (5) is violated for some bidders i and j. However, (5) is not sufficient because we may be able to lower the value of some package won in $w_u(N, X \setminus \{x_i^*\})$ or $w_u(N \setminus \{j\}, X)$ and flip the inequality in (5), thereby creating an opportunity for further revenue growth.

The necessity of condition (5) already rules out most non-additive valuations. Since the Vickrey mechanism treats non-participating bidders as if they participated with all zero values, any violation of bidder revenue monotonicity is also a violation of revenue monotonicity. Therefore, Theorem 1 tells us that we must operate within the set of substitutable valuations. If we restrict the set of substitutable valuations so that each bidder can report a single, fixed, positive value α for at most two goods and those goods must be strict substitutes (not additive), then the following profile violates (5) for $i = 1$, $x_i^* = A$, and $j = 2$.

	A	B	C
Bidder 1	α	α	0
Bidder 2	0	α	α
Bidder 3	0	0	α

We cannot restrict bidders any further in the number of goods or values that they can report without destroying strict substitutability. So, in a setting where the set of possible valuations that one bidder can report does not depend on what other bidders have reported, only additive valuation profiles preserve revenue monotonicity. If we do not allow non-additive value reports, we have eliminated the need for a package auction. Restricting to additive valuations is equivalent to selling each item in a separate auction.

3.2. Bidder-Optimal Core-Selecting Mechanisms

It was once thought that minimum-revenue core-selecting mechanisms were revenue monotonic. (See Day and Milgrom (2008).) In a minimum-revenue core-selecting auction, revenue monotonicity implies bidder revenue monotonicity because a non-participating bidder is treated the same as if they participated with all zero values. However, Lamy (2010) demonstrates that bidder revenue monotonicity, and hence also revenue monotonicity, do not hold in core-selecting auctions that always choose bidder-optimal allocations. This theorem applies to minimum-revenue core-selecting mechanisms because they always choose bidder-optimal allocations.

Theorem 2 (Lamy (2010)). *With at least three goods for sale, there is no bidder-revenue-monotonic core-selecting mechanism that always selects a bidder-optimal imputation.*

Corollary 1. *With at least three goods for sale, there is no revenue-monotonic core-selecting mechanism that always selects a bidder-optimal imputation.*

Corollary 2. *With at least three goods for sale, minimum-revenue core-selecting auctions are neither revenue monotonic nor bidder revenue monotonic.*

Lamy (2010) also gives a sufficient condition for bidder revenue monotonicity of minimum-revenue core-selecting auctions:

Theorem 3 (Lamy (2010)). *When there are only two goods for sale, minimum-revenue core-selecting auctions are bidder revenue monotonic.*

The example in Section 3.1 shows, however, that even with only two goods for sale, minimum-revenue core-selecting auctions are not revenue monotonic. That example deals with Vickrey payoffs, but those Vickrey payoffs correspond with the minimum-revenue core because the valuations are substitutable.

Lamy's theorem restricts the number of goods in the auction. We can allow any number of goods and still achieve bidder revenue monotonicity if we instead restrict the possible valuation profiles. If it is possible for the entering bidder to receive his Vickrey payoff in some core imputation that gives the seller his minimum core revenue, then the seller's minimum core revenue weakly increases.

Proposition 2. *Let N and $\tilde{N} = N \cup h$ be sets of bidders and u and $\tilde{u} = (u, u_h)$ their respective valuation profiles. If $\exists \pi(\tilde{u}) \in Core(\tilde{N}, \tilde{u})$ such that $\pi_0(\tilde{u}) \in \min\{\pi_0 | \pi \in Core(\tilde{N}, \tilde{u})\}$ and $\pi_h(\tilde{u}) = w_{\tilde{u}}(\tilde{N}) - w_{\tilde{u}}(N)$, then the seller's minimum core revenue is weakly higher if h is present.*

Proof: If h is not present, the minimum-revenue problem is that given in (4):

$$\max_{\pi \geq 0} \left\{ \sum_{i=1}^{n} \pi_i \right\} \text{ subject to } w_u(N) - w_u(N \backslash S) \geq \sum_{i \in S} \pi_i \quad \forall S \subseteq N \text{ and } \pi_0 = w_u(N) - \sum_{i=1}^{n} \pi_i.$$

Denote payoffs from the minimum-revenue core-selecting auction by $\pi(u)$. The payoffs after bidder h enters are denoted by $\pi(\tilde{u})$. By assumption, $\pi_h(\tilde{u}) = w_{\tilde{u}}(\tilde{N}) - w_{\tilde{u}}(\tilde{N} \setminus \{h\}) = w_{\tilde{u}}(\tilde{N}) - w_u(N)$. Therefore,

$$\begin{aligned}
\pi_0(\tilde{u}) &= w_{\tilde{u}}(\tilde{N}) - \pi_h(\tilde{u}) - \sum_{i=1}^{n} \pi_i(\tilde{u}) \\
&= w_{\tilde{u}}(\tilde{N}) - \Big(w_{\tilde{u}}(\tilde{N}) - w_u(N) \Big) - \sum_{i=1}^{n} \pi_i(\tilde{u}) \\
&= w_u(N) - \sum_{i=1}^{n} \pi_i(\tilde{u}) \, .
\end{aligned}$$

The minimum-revenue problem is $\max_{\pi \geq 0} \{\sum_{i=1}^{n} \pi_i\}$ subject to

$$w_{\tilde{u}}(\tilde{N}) - w_{\tilde{u}}(\tilde{N} \setminus S) \geq \sum_{i \in S} \pi_i \quad \forall S \subseteq N \tag{6}$$

$$w_{\tilde{u}}(\tilde{N}) - w_{\tilde{u}}(\tilde{N} \setminus \{S, h\}) \geq \pi_h + \sum_{i \in S} \pi_i \quad \forall S \subseteq N. \tag{7}$$

Since $w_{\tilde{u}}(\tilde{N} \setminus \{S, h\}) = w_u(N \setminus S)$ and $\pi_h(\tilde{u}) = w_{\tilde{u}}(\tilde{N}) - w_u(N)$, (7) reduces to $w_u(N) - w_u(N \setminus S) \geq \sum_{i \in S} \pi_i \quad \forall S \subseteq N$, the same set of set of constraints as in (4). Therefore, the new minimum-revenue problem is the same as the old one, but with the extra constraints in (6). Adding more constraints reduces the maximum value of the payoffs of the bidders in N. Thus, $\sum_{i=1}^{n} \pi_i(\tilde{u}) \leq \sum_{i=1}^{n} \pi_i(u)$ and $\pi_0(\tilde{u}) = w_u(N) - \sum_{i=1}^{n} \pi_i(\tilde{u}) \geq w_u(N) - \sum_{i=1}^{n} \pi_i(u) = \pi_0(u)$. ■

Corollary 3. *If the set of possible valuation profiles includes only those for which the coalitional values are bidder submodular, then any minimum-revenue core-selecting auction is bidder revenue monotonic.*

Proof: Ausubel and Milgrom (2006) prove that coalitional values are bidder submodular if and only if the Vickrey imputation $\pi^v \in Core(S, u)$ for all $S \subseteq N$. By Proposition 2, minimum-revenue core-selecting auctions are bidder revenue monotonic whenever the Vickrey imputation is in the core for all $S \subseteq N$. ■

The set of valuation profiles that includes only those for which the coalitional values are bidder submodular is larger than the set of substitutable valuations. For example, the following valuation profile is bidder submodular even though bidder 3's values are not substitutable. (Values for any packages not shown are assumed to be additive.)

	A	B	AB	C
Bidder 1	4	0	4	2
Bidder 2	0	4	4	2
Bidder 3	0	0	6	0

However, without substitutable valuations, bidder submodularity requires coordination between different bidders' valuations. In other words, we cannot just choose all bidder valuations from the same set so that $u \in U^n$. We must specify $u \in U_1 \times \ldots \times U_n$ so that complementarities in values are only present when other bidders' valuations cancel out any potential submodularity problems, such as in the example.

Corollary 4. *For any valuation profile u, there exists a payoff vector in the bidder-optimal frontier of $Core(\tilde{N}, (u, u_h))$ where h receives his Vickrey payoff and the seller's payoff is (weakly) larger than his minimum payoff in $Core(N, u)$.*

Proof: Even if giving bidder h his Vickrey payoff does not give the seller his minimum core revenue, the imputation found in the proof of Proposition 2 is in the core and bidder optimal, since you cannot give any other bidder more without giving bidder h strictly less. ■

Unfortunately, this corollary does not guarantee bidder revenue monotonicity in bidder-optimal core-selecting mechanisms. Lamy's proof of Theorem 3 adds two bidders at once. It may not be possible to give two bidders their combined Vickrey payoff because together they may add more value than the sum of what they add individually: $w(N \cup \{h, l\}) - w(N) > w(N \cup \{h, l\}) - w(N \cup \{h\}) + w(N \cup \{h, l\}) - w(N \cup \{l\})$. Then there is no guarantee that there exists a point in the bidder-optimal frontier where the seller's revenue is at least as great as before the new bidders entered.

4. Revenue-Monotonic Core-Selecting Mechanisms

Section 3 reveals the revenue monotonicity problems encountered by all bidder-optimal core-selecting mechanisms, including minimum-revenue ones. We do not, however, have to give up the core to achieve revenue monotonicity. Section 4.1 gives some simple core-selecting mechanisms that are both revenue monotonic and bidder revenue monotonic. Section 4.2 introduces maximin-revenue mechanisms, which are bidder optimal within the set of revenue-monotonic core-selecting mechanisms, minimize possible gains from untruthful reports, and provide optimal marginal incentives.

4.1. Some Simple Monotonic Core-Selecting Mechanisms

Seller Takes All. The auction with outcome $\pi_0(u) = w - u(N)$ and $\pi_i(u) = 0$ for all $i \in N$, or equivalently, $x(u) \in \arg\max_{x \in X} \left\{ \sum_{i \in N} u_i(x_i) \right\}$ and $p_i(u) = u_i(x_i(u))$ for all $i \in N$, is always in the core, as discussed in Section 2. The seller's revenue is the value of the grand coalition, $w_u(N) = \max_{x \in X} \left\{ \sum_{i \in N} u_i(x_i) \right\}$, which is increasing in both bidders and values. However, this auction never gives bidders strictly positive profits, which in turn gives bidders either no incentive to participate or a large incentive to lie about their valuations.

Functions of $w_u(N \setminus \{i\})$. Revenue $\pi_0(u) = w_u(N \setminus \{i\})$ is always part of a core imputation for any $i \in N$. For example, it is part of the imputation $\pi_0(u) = w_u(N \setminus \{i\})$, $\pi_i(u) = w_u(N) - w_u(N \setminus \{i\})$, and $\pi_j(u) = 0$ for all $j \neq i$. (This imputation was discussed in Section 2.) Of course, there are many other ways to split the remaining surplus, $w_u(N) - w_u(N \setminus \{i\})$, between the bidders and still have an imputation that lies in the core. The value of the coalition of any $n-1$ bidders, $w_u(N \setminus \{i\}) = \max_{x \in X} \left\{ \sum_{j \in N \setminus \{i\}} u_j(x_j) \right\}$, is increasing in both the set of bidders and the reported values, so the seller's revenue is both bidder revenue monotonic and revenue monotonic.

Since both $\max_{i \in N} \{w_u(N \setminus \{i\})\}$ and $\min_{i \in N} \{w_u(N \setminus \{i\})\}$ are possible core revenues, so is any linear combination of the values $\{w_u(N \setminus \{i\})\}_{i \in N}$. This follows from the convexity of the core. In particular, $\pi_0(u) = \min_{i \in N} \{w_u(N \setminus \{i\})\}$ gives the seller the lowest revenue of any such linear combination. This payoff is both bidder revenue monotonic and revenue monotonic because, since each $w_u(N \setminus \{i\})$ is increasing in bidders and values, so is the minimum over the set of these coalitional values.

Though the revenue $\pi_0(u) = \min_{i \in N} \{w_u(N \setminus \{i\})\} \leq w_u(N)$, it can be much closer to the seller's maximum revenue $w_u(N)$ than to the seller's minimum revenue. For example, when Vickrey payoffs lie in the core, the seller receives $\sum_{i \in N \setminus \{m\}} (w_u(N) - w_u(N \setminus \{i\}))$ more than his Vickrey payoff, where $m = \arg\min_{i \in N} \{w_u(N \setminus \{i\})\}$. Moreover, the revenue $\pi_0(u) = \min_{i \in N} \{w_u(N \setminus \{i\})\}$ can be strictly larger than necessary for revenue monotonicity. Consider the following valuation profile (where values for packages not shown are assumed to be additive):

	A	B	AB	C	D
Bidder 1	**3**	0	3	**3**	1
Bidder 2	0	**3**	3	1	**2**
Bidder 3	0	0	5	0	0

In this example, $\pi_0(u) = \min_{i \in N} \{w_u(N \setminus \{i\})\} = w_u(N \setminus \{1\}) = 8$. The Vickrey payoffs are in the core so the minimum core revenue is the seller's Vickrey payoff, which is 6. To guarantee revenue monotonicity, we need to ensure that decreasing the valuation profile will not force the seller to receive a higher payoff in order to stay within the core. In other words, we need $\pi_0(u) \geq \min \{\pi_0 | \pi = (\pi_0, \ldots, \pi_n) \in Core(u')\}$ for all $u' \leq u$. The highest minimum core revenue generated by any lower valuation profile is 7, which comes from the profile:

	A	B	AB	C	D
Bidder 1	**2**	0	2	**3**	1
Bidder 2	0	**3**	3	1	**2**
Bidder 3	0	0	5	0	0

This example motivates the class of core-selecting auctions discussed in Sections 4.2 and 4.3.

4.2. Maximin-Revenue Core-Selecting Mechanisms

Now we propose a class of core-selecting mechanisms that are both revenue monotonic and bidder revenue monotonic.

Definition 10. *A core-selecting mechanism is maximin-revenue if, for any valuation profile $u = (u_1, \ldots, u_n)$, the seller receives the maximum of all minimum core revenues for all lower valuation profiles:*

$$\pi_0(u) = \max_{u' \leq u} \{\min \{\pi_0 | \pi = (\pi_0, \ldots, \pi_n) \in Core(u')\}\}.$$

Proposition 3. *Any maximin-revenue core-selecting mechanism is revenue monotonic and bidder revenue monotonic.*

Proof: For $\hat{u} \leq u$, any maximin-revenue core-selecting mechanism chooses

$$\begin{aligned}\pi_0(\hat{u}) &= \max_{u' \leq \hat{u}} \{\min \{\pi_0 | \pi = (\pi_0, \ldots, \pi_n) \in Core(u')\}\} \\ &\leq \max_{u' \leq u} \{\min \{\pi_0 | \pi = (\pi_0, \ldots, \pi_n) \in Core(u')\}\} \\ &= \pi_0(u),\end{aligned}$$

where the inequality follows from the fact that we are maximizing over a larger set because $\hat{u} \leq u$. Thus, any maximin-revenue core-selecting mechanism is revenue monotonic. The minimum revenue in the core is the same when a bidder does not participate as when he does participate with values of zero for every possible package. Set $\hat{u}_i(x_i) = 0$ for all $x_i \in X_i$ and $\hat{u}_j(x_j) = u_j(x_j)$ for all $j \neq i$ and all $x_j \in X_j$. Then $\pi_0(u_{-i}) = \pi_0(\hat{u}) \leq \pi_0(u)$ by the previous argument because $\hat{u} \leq u$. Thus, any maximin-revenue core-selecting mechanism is also bidder revenue monotonic. ■

Finding the maximin core revenue is a daunting task. The set $\{u' | u' \leq u\}$ is infinite and every minimization problem using one of the infinite members of the set requires computing bounds and then solving a system of linear inequalities. However, as it turns out, we do not need to compute every core constraint and then run a linear program on the system of inequalities to determine the minimum core revenue at each lower valuation profile. Theorem 4 shows that we only need to compute the Vickrey payoffs at each lower valuation profile. Not only does this save us from running the linear program, it means we never have to compute $w(N \setminus S)$ for any $S \subseteq N$ such that $|S| > 1$. When $n = |N|$ is large, this eliminates a huge number of calculations.

Moreover, Theorem 4 shows that the core is unnecessary if you already require revenue monotonicity. One benefit of the core is that it eliminates some of the low revenue problems that the Vickrey mechanism encounters when items are complements. However, at least at its bidder optimal frontier, it does not solve problems of non-monotonicities. Theorem 4 establishes that a revenue-monotonic mechanism can be derived directly from the Vickrey payoffs.

Definition 11. *We say that a mechanism is maximum Vickrey if it assigns the seller the maximum of his Vickrey payoffs under all lower valuation profiles:*

$$\pi_0(u) = \max_{u' \leq u} \{\pi_0^v(u')\}.$$

Theorem 4. *For every maximin-revenue core-selecting mechanism, there is an equivalent maximum Vickrey mechanism.*

The proof makes use of the following two lemmas. They show that the minimum core revenue of any valuation profile $u = (u_1, \ldots, u_n)$ can be achieved using the seller's Vickrey payoff at some lower valuation profile.

Lemma 1. *If* $x^* \in \arg\max_{x\in X} \left\{\sum_{i\in N} u_i(x_i)\right\}$*, then*

$$x^* \in \arg\max_{x\in X} \left\{ \sum_{i\in N\setminus\{j\}} u_i(x_i) + \hat{u}_j(x_j) \right\}$$

where $\hat{u}_j(x_j) = \max\{u_j(x_j) - (w_u(N) - w_u(N \setminus \{j\})), 0\}$.

Proof: Under $\hat{u} = (u_{-j}, \hat{u}_j)$, if no package is assigned to bidder j, then the maximum surplus is $\max_{x\in X} \left\{\sum_{i\in N\setminus\{j\}} u_i(x_i)\right\} = w_u(N \setminus \{j\})$. If some package is assigned to bidder j, the maximum surplus is still no more than $w_u(N\setminus\{j\})$ because either $\hat{u}_j(x_j) = 0$ or we solve

$$\begin{aligned}
&\max_{\{x\in X|u_j(x_j)\geq w_u(N)-w_u(N\setminus\{j\})\}} \sum_{i\in N\setminus\{j\}} u_i(x_i) + \hat{u}_j(x_j) \\
&= \max_{\{x\in X|u_j(x_j)\geq w_u(N)-w_u(N\setminus\{j\})\}} \sum_{i\in N} u_i(x_i) - (w_u(N) - w_u(N \setminus \{j\})) \\
&\leq \max_{\{x\in X\}} \sum_{i\in N} u_i(x_i) - (w_u(N) - w_u(N \setminus \{j\})) \\
&= w_u(N) - (w_u(N) - w_u(N \setminus \{j\})) \\
&= w_u(N \setminus \{j\}).
\end{aligned}$$

We know that x^* achieves a surplus of $w_u(N \setminus \{j\})$ under $\hat{u} = (u_{-j}, \hat{u}_j)$ and, therefore, $x^* \in \arg\max_{x\in X} \left\{\sum_{i\in N\setminus\{j\}} u_i(x_i) + \hat{u}_j(x_j)\right\}$. ■

Lemma 2. *Let* $u = (u_1, \ldots, u_n)$ *be such that the Vickrey payoffs* $\pi_i^v(u) = w_u(N) - w_u(N \setminus \{i\})$ *for all* $i \in N$ *and* $\pi_0^v(u) = w_u(N) - \sum_{i \in N} \pi_i^v(u)$ *are not in the core. Choose any efficient allocation* $x^* \in \arg\max_{x \in X} \sum_{i \in N} u_i(x_i)$. *Then there exists a lower valuation profile* $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$ *such that:*

(1) $x^* \in \arg\max_{x \in X} \sum_{i \in N} \hat{u}_i(x_i)$

(2) $\pi^v(\hat{u}) \in Core(N, \hat{u})$

(3) *If*

- $\pi_i = u_i(x_i^*) - \hat{u}_i(x_i^*) + \pi_i^v(\hat{u})$ *for all* $i \in N$ *and*
- $\pi_0 = w_u(N) - \sum_{i \in N} \pi_i = w_{\hat{u}}(N) - \sum_{i \in N} \pi_i^v(\hat{u}) = \pi_0^v(\hat{u})$,

then $\pi = (\pi_0, \ldots, \pi_n) \in Core(N, u)$.

Proof: The Vickrey payoffs are in the core with respect to u if and only if for all $S \subseteq N$,

$$\sum_{i \in S} \pi_i^v(u) = \sum_{i \in S} [w_u(N) - w_u(N \setminus \{i\})] \leq w_u(N) - w_u(N \setminus S). \quad (8)$$

If the Vickrey payoffs are not in the core, then choose the smallest set $S \subseteq N$ (or any one of the smallest sets if there are multiple ones of the same size) such that (8) does not hold. Relabel the bidders so that $S = \{1, \ldots, k\}$. Note that $k \geq 2$ because the inequality holds trivially for $k = 1$. Also notice that every $s \in S$ is a winning bidder and has a positive Vickrey payoff $w_u(N) - w_u(N \setminus \{s\}) > 0$. Otherwise, if $w_u(N) - w_u(N \setminus \{s\}) = 0$, then the set $S \setminus \{s\}$ is strictly smaller than S and violates (8) because $\sum_{i \in S \setminus \{s\}} \pi_i^v(u) = \sum_{i \in S} \pi_i^v(u) > w_u(N) - w_u(N \setminus S) \geq w_u(N) - w_u(N \setminus S \cup \{s\})$. We have $\sum_{i \in S} [w(N) - w(N \setminus \{i\})] > w(N) - w(N \setminus S)$. Rearranging the inequality leads to

$$\sum_{i=1}^{k-1} [w(N) - w(N \setminus \{i\})] > w(N \setminus \{k\}) - w(N \setminus S)\,. \quad (9)$$

Start with bidder 1 and decrease his value for each possible package by his Vickrey payoff so that $\hat{u}_1(x_1) = \max\{u_1(x_1) - (w_u(N) - w_u(N \setminus \{1\})), 0\}$. Next, decrease bidder 2's value for each possible package by his Vickrey payoff under the new valuation profile resulting from the change in bidder 1's value function. Thus, $\hat{u}_2(x_2) = \max\{u_2(x_2) -$

$(w_{(\hat{u}_1,u_{-1})}(N) - w_{(\hat{u}_1,u_{-1})}(N \setminus \{2\})), 0\}$. Repeat this process with each bidder $j \leq k-1$ so that

$$\hat{u}_j(x_j) = \max\{u_j(x_j) - (w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N) - w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N \setminus \{j\})), 0\}\,.$$

By Lemma 1, $x^* \in \arg\max_{x \in X} \sum_{i \in N} \hat{u}_i(x_i)$. The proof of Lemma 1 also shows that when we truncate any bidder j's value function by his Vickrey payoff, his new Vickrey payoff is zero, i.e.,

$$\begin{aligned}
&\pi_j^v(\hat{u}_1, \ldots, \hat{u}_j, u_{j+1}, \ldots, u_n) \\
&\quad = w_{(\hat{u}_1,\ldots,\hat{u}_j,u_{j+1},\ldots,u_n)}(N) - w_{(\hat{u}_1,\ldots,\hat{u}_j,u_{j+1},\ldots,u_n)}(N \setminus \{j\}) \\
&\quad = 0\,.
\end{aligned}$$

Also, truncating bidder j's value function by any constant, say α, can never cause any other bidder's Vickrey payoff to increase. Since the value of every package is reduced by the same amount, any $w(T)$ can decrease by at most α, which occurs if bidder j is winning some package in T and the value that winning package adds to $w(T \setminus \{j\})$ is at least α. The value of the grand coalition, $w(N)$, always decreases by the full amount α. Therefore, $w(N) - w(N \setminus \{i\})$ cannot increase. This means that once we reduce bidder j's Vickrey payoff to zero, it stays zero under all further modifications. Thus, the new Vickrey payoffs satisfy constraint (8) with respect to $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_{k-1}, u_k, \ldots, u_n)$:

$$\sum_{i \in S} \pi_i^v(\hat{u}) = 0 + \pi_k^v(\hat{u}) = w_{\hat{u}}(N) - w_{\hat{u}}(N \setminus \{k\}) \leq w_{\hat{u}}(N) - w_{\hat{u}}(N \setminus S)$$

because $\{k\} \subseteq S$. The new Vickrey payoffs also satisfy constraint (8) with respect to the original valuation profile, u. For any $j \leq k-1$,

$$\begin{aligned}
\pi_j &= u_j(x_j^*) - \hat{u}_j(x_j^*) + \pi_j^v(\hat{u}) \\
&= u_j(x_j^*) - \left[u_j(x_j^*) - (w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N) - w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N \setminus \{j\}))\right] + 0 \\
&= w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N) - w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N \setminus \{j\})\,.
\end{aligned}$$

Also,

$$
\begin{aligned}
\pi_k &= u_k(x_k^*) - \hat{u}_k(x_k^*) + \pi_k^v(\hat{u}) \\
&= u_k(x_k^*) - u_k(x_k^*) + w_{\hat{u}}(N) - w_{\hat{u}}(N \setminus \{k\}) \\
&= w_{\hat{u}}(N) - w_{\hat{u}}(N \setminus \{k\})\,.
\end{aligned}
$$

We know that at each step we have decreased the value of the grand coalition by

$$
w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N) - w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N \setminus \{j\})\,,
$$

so

$$
\begin{aligned}
w_{\hat{u}}(N) &= w_u(N) - \sum_{j=1}^{k-1} \left[w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N) - w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N \setminus \{j\}) \right] \\
&= w_u(N) - \sum_{j=1}^{k-1} \pi_j\,.
\end{aligned}
$$

Moreover, because at each step we have only decreased bidder j's values, we must have

$$
\begin{aligned}
w_{\hat{u}}(N \setminus \{k\}) &\geq w_{(\hat{u}_1,\ldots,\hat{u}_{k-2},u_{k-1},\ldots,u_n)}(N \setminus \{k-1,k\}) \\
&\geq \ldots \geq w_{(\hat{u}_1,u_{-1})}(N \setminus \{2,\ldots,k\}) \geq w_u(N \setminus S)\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{j=1}^{k-1} \pi_j + \pi_k &= \sum_{j=1}^{k-1} \pi_j + w_{\hat{u}}(N) - w_{\hat{u}}(N \setminus \{k\}) \\
&= \sum_{j=1}^{k-1} \pi_j + w_u(N) - \sum_{j=1}^{k-1} \pi_j - w_{\hat{u}}(N \setminus \{k\}) \\
&\leq w_u(N) - w_u(N \setminus S)
\end{aligned}
$$

and the inequality in (8) is now satisfied. Furthermore, the seller receives his Vickrey payoff under the new profile because

$$
\begin{aligned}
\pi_0 &= w_u(N) - \sum_{i \in N} \pi_i \\
&= w_u(N) - \sum_{i \in N} (\hat{u}_i(x_i^*) - \hat{u}_i(x_i^*) + \pi_i^v(\hat{u})) \\
&= w_u(N) - \sum_{j=1}^{k-1} \left[w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N) - w_{(\hat{u}_1,\ldots,\hat{u}_{j-1},u_j,\ldots,u_n)}(N \setminus \{j\}) \right] - \sum_{i \in N} \pi_i^v(\hat{u}) \\
&= w_{\hat{u}}(N) - \sum_{i \in N} \pi_i^v(\hat{u}) \\
&= \pi_0^v(\hat{u}) \,.
\end{aligned}
$$

So far we have proved parts 1, 2, and 3 of the theorem when there was only a single $S \subseteq N$ for which (8) was violated. Of course, after the valuation profile change to $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_{k-1}, u_k, \ldots, u_n)$, there may be remaining sets $S \subseteq N$ that violate (8). If so, we can repeat the process until we correct all such violations. Note that once we have applied the process to some bidder, he can never again be part of a problem set because his Vickrey payoff is zero forever afterwards. There are a finite number of bidders, so this process will eventually end. When it does, the original efficient allocation will remain efficient, the new valuation profile will have Vickrey payoffs in its core, and the modified payoffs described in part 3 will be in the core with respect to the original valuation profile. ■

Proof of Theorem 4: By Lemma 2, the minimum core revenue obtained under any valuation profile can be obtained as the seller's Vickrey payoff under some lower valuation profile. That means that, for a given valuation profile, any maximum Vickrey mechanism results in at least as much revenue for the seller as any maximin-revenue core-selecting mechanism. A maximum Vickrey mechanism cannot result in revenue higher than a maximin-revenue core-selecting mechanism because $\pi_0^v(u) \leq \min\{\pi_0 \,|\, \pi = (\pi_0, \ldots, \pi_n) \in Core(u)\}$ for all valuation profiles u. Thus, the seller receives the same revenue in both classes of mechanisms and we have the same remaining surplus to divide between the bidders in both cases. A maximum Vickrey mechanism that employs the same rule in selecting the bidders' payoffs will be equivalent to a given maximin-revenue core-selecting mechanism. ■

Theorem 4 narrows down the possibilities that we must consider when looking over all lower valuations for the maximin core revenue. The method of reducing valuations used in the proof does not, however, guarantee the maximin revenue. That is, we must sometimes lower individual package values, rather than just truncating by a constant, to find the maximin core revenue. This concept is confirmed by the necessary condition for revenue monotonicity in Proposition 3, in which one must evaluate not the value added by each winning bidder but rather the value added by each winning bidder's ability to receive his winning package. Given Theorem 4, this necessary condition becomes a guide to finding the maximin revenue.

To further narrow the search, Theorem 5 tells us that we only need to consider lower valuation profiles for which the efficient allocation remains the same.

Theorem 5. *Given any valuation profile $u = (u_1, \ldots, u_n)$ and any efficient allocation $x^* \in \arg\max_{x \in X} \sum_{i \in N} u_i(x_i)$, the maximum Vickrey revenue occurs at a lower valuation profile $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$ such that $x^* \in \arg\max_{x \in X} \sum_{i \in N} \hat{u}_i(x_i)$.*

Proof: Suppose not. Then there exists a profile $u' = (u'_1, \ldots, u'_n) \leq u$ such that $x^* \notin \arg\max_{x \in X} \sum_{i \in N} u'_i(x_i)$ which gives strictly higher revenue than any $\hat{u} \leq u$ with $x^* \in \arg\max_{x \in X} \sum_{i \in N} \hat{u}_i(x_i)$. Let u'' be the profile such that $u''_i(x^*_i) = u_i(x^*_i)$ and $u''_i(x_i) = u'_i(x_i)$ for all $x_i \neq x^*_i$ and for all $i \in N$. The seller's Vickrey payoff under u'' is

$$\begin{aligned}
\pi^v_0(u'') &= w_{u''}(N) - \sum_{i \in N} \pi^v_i(u'') \\
&= w_{u''}(N) - \sum_{i \in N} (w_{u''}(N) - w_{u''}(N \setminus \{i\})) \\
&= \sum_{i \in N} w_{u''}(N \setminus \{i\}) - (n-1) w_{u''}(N) \,.
\end{aligned}$$

Let $\gamma = w_{u''}(N) - w_{u''}(N, X \setminus x^*)$ where $w_{u''}(N, X \setminus x^*)$ is the value of the grand coalition when the feasible set is restricted to

$$\{x \in X \,|\, x_i^* \nsubseteq x_i \,\forall\, i \in N \text{ with } x_i^* \neq \emptyset\}\,.$$

Thus, γ is the value added by the ability of each bidder to win his efficient package. To change the efficient allocation, the value of the current efficient allocation $\sum_{i \in N} u_i''(x_i^*)$ must decrease by strictly more than γ. However, notice that when we make the value of the efficient allocation decrease by moving from u'' to u', $w_{u''}(N)$ drops by exactly γ and no more, increasing the seller's payoff by $(n-1)\gamma$. If we had decreased $\sum_{i \in N} u_i''(x_i^*)$ by exactly γ, then $w_{u''}(N)$ would still have dropped by exactly γ. Now consider changes in $w_{u''}(N \setminus \{i\})$. These coalitional values can only decrease when the value of the efficient allocation is reduced. They are monotonic in the bidders' package values, so the more we decrease the value of the old efficient allocation, the more we reduce each $w_{u''}(N \setminus \{i\})$ and, thereby, the more we reduce the seller's revenue. Therefore, lowering the value of the efficient allocation by strictly more than γ can only hurt the seller's revenue. Thus, the seller gets at least as much revenue when the value of the efficient allocation drops by exactly γ and the original efficient allocation remains efficient. ■

4.3. Incentives in Maximin-Revenue Mechanisms

Maximin-revenue core-selecting mechanisms are harder to compute than some of the other examples of monotonic mechanisms described in section 4.1. However, they provide "better" bidder incentives than those simpler mechanisms. This is documented in the following theorems but, first, we must describe a metric for comparing the incentives provided by different core-selecting mechanisms. Let P_{CS} denote the set of all core-selecting mechanisms. For any $p \in P_{CS}$, let $\varepsilon_i^p(u)$ denote bidder i's maximum gain from deviating from truthful reporting (when other bidders report their values truthfully).

Definition 12 (Day and Milgrom (2008)). *A core-selecting auction $p \in P \subseteq P_{CS}$ provides* optimal incentives *within the set of auctions P if for all valuation profiles u*

there does not exist any $p' \in P$ such that $\varepsilon_i^{p'}(u) \leq \varepsilon_i^{p}(u)$ for all $i \in N$ and the inequality is strict for some $i \in N$.

Theorem 6. *Maximin-revenue core-selecting mechanisms give the seller the minimum revenue out of all revenue-monotonic core-selecting mechanisms, i.e. they are bidder-optimal revenue-monotonic core-selecting mechanisms.*

Proof: By construction, if any core-selecting mechanism were to give the seller a strictly lower payoff at any profile $u = (u_1, \dots, u_n)$, then there is some profile $\hat{u} \leq u$ such that the minimum-revenue in the core of $\hat{u}$ is strictly higher than the seller's revenue under u. Since we are talking about core-selecting mechanisms, this means $\pi_0(\hat{u}) > \pi_0(u)$ and the mechanism is not revenue monotonic. ■

Corollary 5. *Maximin-revenue core-selecting auctions give optimal incentives within the set of all revenue-monotonic core-selecting auctions.*

Proof: As discussed in Section 2, bidder i's largest payoff in the core is his Vickrey payoff. Moreover, by Theorem 2 of Day and Milgrom (2008), bidder i's true Vickrey payoff is the maximum payoff he can receive in any core-selecting mechanism, no matter what valuation function he reports. His maximum gain from deviating from truthful reporting is therefore the difference between his Vickrey payoff and the actual payoff he receives. So a core-selecting mechanism provides optimal incentives within a set P precisely when it is bidder optimal within that set. ■

The corollary to Theorem 6 says that maximin-revenue core-selecting auctions minimize the maximum possible gains from reporting false valuations. The mechanisms in Section 4.1 do strictly worse in this respect because they are not bidder optimal within the set of revenue-monotonic core-selecting auctions. The fact that they are not bidder optimal can be seen in the example given at the end of Section 4.1. Given any core-selecting auction that gives the seller strictly more revenue than the maximum revenue, bidders strictly prefer the maximin-revenue auction that assigns the same payoffs as the given auction except that it distributes to some eligible bidders the extra revenue not given to the seller.

Maximum possible gains from deviations from truthful reporting are not the only way to measure the incentives provided by a core-selecting mechanism. When bidders' valuations are private information, bidder i does not know the exact value of $\varepsilon_i^p(u)$. In this situation, attempts to capture the expected maximum gain from deviation are risky. For this reason, Erdil and Klemperer (2010) suggest evaluating mechanisms by their marginal incentives.

Definition 13. *Let* $\Delta_i^p(u,\delta) = \max_{\{\hat{u}_i \mid |u_i(x_i)-\hat{u}_i(x_i)|<\delta\}} [\pi_i^p(\hat{u}_i, u_{-i}) - \pi_i^p(u)]$ *denote the maximum gain from a deviation of size* δ*. Then bidder* i*'s* marginal incentive *to deviate from truthful reporting (when other bidders report truthfully) is* $\Delta_i^p(u) = \max\left\{\lim_{\delta\to 0} \frac{\Delta_i^p(u,\delta)}{\delta}, 0\right\}$.

Definition 14. *Take any two core-selecting auctions* $p, \hat{p} \in P_{CS}$*. We say that* $\hat{p}$ *provides* better marginal incentives *than* p *if* $\Delta_i^{\hat{p}}(u) \leq \Delta_i^p(u)$ *for all* $i \in N$ *and all possible valuation profiles* u*. If the inequality is strict for some* $i \in N$ *and some* u*, we say that* $\hat{p}$ *provides* strictly better marginal incentives *than* p*.*

Not only do maximin-revenue core-selecting mechanisms provide optimal incentives, they also provide strictly better marginal incentives than any other revenue-monotonic core-selecting mechanisms. The intuition is that the maximin revenue is, in general, not affected by small changes in valuations since some strictly lower valuation profile usually determines the seller's revenue.

Theorem 7. *Given any revenue-monotonic core-selecting auction that is not maximin revenue, there exists a maximin-revenue core-selecting auction that provides strictly better marginal incentives.*

Proof: There are two parts to any revenue-monotonic core-selecting mechanism – the choice of the seller's revenue and the allocation of the remaining surplus between the bidders. Given some revenue rule $\pi_0(u)$, any core-selecting auction must choose $p(u)$ to satisfy

- $\sum_{i \in N} p_i(u) = \pi_0(u)$ and

- $\sum_{i\in S} u_i(x_i^*) \geq \sum_{i\in S} p_i(u) \geq \sum_{i\in S} u_i(x_i^*) - w_u(N) + w_u(N \setminus S)$ for all $S \subseteq N$.

The first condition is equivalent to

- $\pi_0(u) = w_u(N) - \sum_{i\in N} \pi_i(u)$ and the second is equivalent to

- $0 \leq \sum_{i\in S} \pi_i(u) \leq w_u(N) - w_u(N \setminus S)$.

Thus, a bidder has two ways of influencing his payment – changing the seller's revenue or changing the upper and lower bounds constraining his payment.

Take any $p \in P_{CS}$ that is revenue monotonic but not maximin revenue and let p' denote the maximin-revenue auction that (1) chooses the same efficient allocation as p if there are multiple such allocations to choose from, and (2) given any revenue for the seller, uses the same rule to allocate the remaining surplus among the bidders. We show that any marginal change has less effect on the choice of the seller's revenue under p' and that p' makes it less likely that the bounds constraining the payments are binding. Therefore, p' provides better marginal incentives for every bidder.

Bidders prefer lower seller revenue. In a maximin-revenue auction, the only way to reduce the seller's revenue is to move the valuation profile below the smallest $\hat{u} \leq u$ such that $\hat{u} \in \arg\max_{u' \leq u} \{\min \{\pi_0 | \pi = (\pi_0, \ldots, \pi_n) \in Core(u')\}\}$. When the smallest such $\hat{u}$ is strictly less than u, no marginal change in a bidder's reported values can effect the seller's revenue. If the smallest such $\hat{u}$ is equal to u, then $\pi^v(u) \in Core(N, u)$ by Lemma 2 and the only way to give the seller his Vickrey payoff is to also give each bidder his Vickrey payoff. So the outcome of p' is $\pi^v(u)$. The Vickrey mechanism is strategy-proof, so every bidder prefers $\pi_i^v(u)$ to the payoff he receives in the Vickrey mechanism when he lies about his value and reports some $\hat{u}_i$. That Vickrey payoff is the highest he can possibly achieve in $Core(N, (\hat{u}_i, u_{-i}))$ and so he has no incentive to lie. Therefore, under p', marginal incentives to change the seller's revenue are zero and hence, are weakly lower than the marginal incentives to do so under p. Bidders prefer to lower both the upper- and lower-bound constraints to reduce the level of their feasible payments. First note that bidder $i \in S$ cannot change the lower bound on any constraint that affects his payment because $\sum_{i\in S} u_i(x_i^*) - w_u(N) + w_u(N \setminus S) = -\sum_{j\notin S} u_j(x_j^*) + w_u(N \setminus S)$, which does not depend on his reported values. He can decrease the upper bound and has a positive marginal incentive to do so if and only if that constraint is binding. For any valuation profile u, Theorem 6 tells us that $\pi_0^{p'}(u) \leq \pi_0^{p}(u)$. This inequality is sometimes strict because p is not maximin revenue by assumption. Notice that, given an allocation rule for dividing the remaining surplus between the bidders, whenever the seller's revenue is higher, the upper-bound constraints are more likely to be binding. We have chosen p' to use the same allocation rule as p and therefore, p provides strictly worse marginal incentives to deviate in ways that lower the upper-bound constraints.

Combining the two types of possible marginal gains from lying, we find that p' provides strictly better marginal incentives than p. ■

Theorem 7 shows that maximin revenue core-selecting auctions dominate all other revenue-monotonic core-selecting auctions in terms of marginal incentives for lying. Therefore, maximin-revenue mechanisms win in terms of both maximum and marginal gains to deviations from truth-telling.

Though Theorem 7 does not directly rank auctions within the set of maximin-revenue mechanisms, its proof demonstrates that the best maximin-revenue auction in terms of marginal incentives will prevent upper-bound constraints from binding whenever possible. This means that we want to keep payments strictly below winning values: $p_i(u) < u_i(x_i^*)$. Consider a reference rule for allocating the remaining surplus among bidders that chooses the $p(u)$ that is closest in Euclidean distance to the origin while still satisfying the necessary constraints.[4] This rule pulls payments towards the origin, alleviating some, but not all since no core-selecting mechanism is strategy-proof, of the marginal incentives for deviations from truthful reporting.

[4] See Erdil and Klemperer (2010) for a discussion of reference rules.

5. Conclusion

This paper demonstrates that revenue- and bidder-revenue-monotonic core-selecting auction mechanisms exist and can have relatively simple rules determining the seller's revenue. Maximin-revenue core-selecting mechanisms require harder calculations, but provide the best marginal and non-marginal incentives for truthful reporting out of all revenue-monotonic core-selecting auctions. This paper provides methods for simplifying the maximin-revenue calculations, including restricting attention to Vickrey payoffs and valuation profiles that have the same efficient allocation.

Within the class of maximin-revenue core-selecting auctions, it is an open question as how to allocate the remaining surplus among the bidders. The allocation rule most certainly has large implications for bidder incentives and equilibria. We suggest the reference rule with reference payments of zero for all bidders, but this rule deserves a more detailed analysis of incentives and equilibria.

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