Unrelated Goods in Package Auctions – Comparing Vickrey and Core-Selecting Outcomes

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Abstract

Package auctions were designed for items that are complements or substitutes, but we consider how these auctions perform in the presence of goods that exhibit no such relation to each other. These unrelated goods give a connection between outcomes of Vickrey and core-selecting auctions: any equilibrium outcome of the Vickrey auction is part of an equilibrium outcome of any bidder-optimal core-selecting auction in a possibly extended setting in which unrelated goods are added. Unrelated goods also lead to surprising revenue implications: a seller always receives at least as much (and in many cases strictly more) revenue when running separate minimum-revenue core-selecting auctions for groups of unrelated goods.

1. Introduction

Combinatorial auctions were designed to allow bidders to express preferences that exhibit substitutability or complementarity between goods. The ability to place bids on packages of items, rather than only on individual items, shields bidders from the risk of winning only one of two goods that are complements or winning multiple substitute goods. This richer message space still allows for bids on individual items, so it doesn't prevent bidders from expressing simpler, additive valuations. This paper considers how different combinatorial auctions – in particular, the Vickrey auction applied to package bidding and core-selecting auctions – treat these *unrelated goods* – those for which all bidders have additive values. One desirable property might be what we define as *outcome additivity*: the assignment and payments determined by the auction in the presence of unrelated goods are the sums of the assignments and payments that result from auctioning the unrelated goods separately. We show that the Vickrey auction satisfies this property while bidder-optimal core-selecting auctions do not. This fact leads to some important differences between these auctions and also some important similarities. Since the outcomes of these auctions change in different ways when unrelated goods are added, unrelated goods can be used to equate their outcomes, thereby highlighting similarities as well as differences.

The Vickrey auction is strategy proof and efficient but also has some undesirable properties that led Day and Milgrom (2008) to propose core-selecting auctions as alternatives. These undesirable properties include low revenues, non-monotonicity of the seller's revenues with changes in bids, incentives for bidders to submit shill bids under false identities, and incentives for bidders to collude (see Ausubel and Milgrom (2006) for a detailed discussion). It is known that revenue non-monotonicities (and, consequently, incentives for collusion) exist in bidder-optimal core-selecting auctions as well (Lamy (2010), Beck and Ott (2009)). We use unrelated goods to

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compare the types of valuation profiles for which these non-monotonicities exist in bidder-optimal core-selecting auctions and the Vickrey auction. For given bids, we show that the Vickrey auction can exhibit non-monotonic revenues while bidder-optimal core-selecting auctions do not and that the reverse is also true. However, we can link *any* setting that leads to non-monotonic revenue in the Vickrey auction with one that gives non-monotonic revenue in all bidder-optimal core-selecting auctions, by adding some unrelated goods. In fact, we demonstrate that, by adding unrelated goods in a particular way, we can take any possible setup of bidders, items, and valuations and make the original Vickrey payoffs in the core. This means that, fixing reported values, for every potential setting that is problematic for the Vickrey auction there will be a setting (with unrelated goods) that is problematic for bidder-optimal core-selecting auctions. We also show that the same conclusion holds in equilibrium. For every setting and equilibrium of the Vickrey auction that is problematic there exists a setting with unrelated goods and an equilibrium of all bidder-optimal core-selecting auctions that is problematic. The maximum number of additional items needed to make the Vickrey payoffs in the core is $\binom{m}{2}$ where m is the number of contributing bidders (those that make a non-zero marginal contribution of value to the rest of the bidders and the seller).

Core-selecting auctions remove incentives for bidders to submit shill bids and meet the revenue standard of the core with respect to the bids. (Of course, in equilibrium they can lead to much lower revenues than those in the core with respect to the true values.) However, they have other, less appealing revenue properties. The payments in any minimum-revenue core-selecting auction (a subset of bidder-optimal core-selecting auctions) are always at least as large and, because outcome additivity does not hold for these auctions, sometimes strictly greater when groups of unrelated goods are auctioned separately. In other words, a seller should not auction unrelated items together if he wants to use a minimum-revenue core-selecting auction. If a seller does not have the ability to chose the auction rules or the items to be auctioned, then he would instead have an incentive to disaggregate and take on a separate selling identity for each group of unrelated items. In this sense, minimum-revenue core-selecting auctions remove the incentives for bidders to submit shills while adding an incentive for sellers to submit shills. However, we describe other core-selecting auctions that do not create these incentives for sellers because they satisfy outcome additivity.

The remainder of this paper is organized as follows. Section 2 sets up the package auction problem, defines the core, and describes the classes of bidder-optimal and minimum-revenue core-selecting auctions. Section 3 defines unrelated goods and outcome additivity and develops the close connection between the bidder-optimal core-selecting auctions and the Vickrey package auction, both for fixed value inputs and in equilibrium. It contains an example to show how the problems from the Vickrey package auction carry over into all bidder-optimal core-selecting auctions. Section 4 discusses the lower revenue that results from combining unrelated goods into one minimum-revenue core-selecting auction and provides a different core-selecting auction as a partial solution. Section 5 concludes.

2. Model

We consider a setting with one seller, whom we denote 0, and a set of bidders $N = \{1, \ldots, n\}$. The seller owns a set of goods $K = \{1, \ldots, k\}$, which he does not value. Each bidder i has private values $v_i : 2^K \to \mathbb{R}_+$ for packages $y \in 2^K$. We normalize $v_i(\emptyset) = 0$ for all $i \in N$ and we denote the vector of these values $v = (v_1, \ldots, v_n)$. Bidders have quasilinear utility, so if bidder i wins bundle y and pays price p_i, then he gets a payoff of $v_i(y) - p_i$. The seller's payoff is the sum of the payments made by the bidders.

Bidders place bids $b = (b_1, \ldots, b_n)$, where the vector of bidder i's bids for all bundles is $b_i = (b_i(y))_{y \in 2^K}$. Let b_S and b_{-S} be the vectors of bids of groups $S \subseteq N$ and $N \setminus S$, respectively.

Let $B = B_1 \times \cdots \times B_n$ where B_i denotes the set of feasible bids for bidder i and assume, unless otherwise noted, that $B_i = \mathbb{R}_+^{2^K}$ with the normalization $b_i(\emptyset) := 0$.

A sealed-bid package auction is a direct mechanism that assigns each bidder a package of goods $x_i(b)$ and a payment $p_i(b, x(b))$ based on bids. The set of feasible assignments of goods $L \subseteq K$ is $X(L) = \{x = (x_1, \ldots, x_n) | x_i \in 2^L, x_i \cap x_j = \emptyset \text{ for all } i \neq j \in N\}$. For ease of notation, we will usually drop the index L when $L = K$. The payment depends on the assignment as well as the bids because when randomizations are used to break ties, the payment will change with the realized assignment $x(b)$. We sometimes write $p(b) \equiv p(b, x(b))$ when the relationship with the assignment is clear.

Most of the auctions we consider are part of the class of core-selecting auctions, which use a concept from cooperative game theory as part of their design. To describe these auctions, we must first define the coalitional function w and the corresponding set of optimal assignments $\hat{X}$:

$$w(b) = \max_{x \in X(K)} \sum_{i \in N} b_i(x_i)$$

$$\hat{X}(b) = \arg\max_{x \in X(K)} \sum_{i \in N} b_i(x_i)$$

We call w the coalitional function because it is the maximum reported value that the coalition of the bidders and the seller can generate by trading. There may be multiple *optimal* (with respect to bids) assignments that achieve this maximum reported value, so $\hat{X}(b)$ need not be a singleton. Note that the optimal assignment need not be (true) value-maximizing. We call any value-maximizing assignment $x \in \hat{X}(v)$ *efficient* because, in our auction games, such an assignment corresponds to efficient payoffs.

To represent the value generated by the coalition of some subset of bidders $S \subseteq N$ and the seller or the value generated using only some subset of the seller's goods $L \subseteq K$, we use the notation:

$$w(b_S^L) = \max_{x \in X(L)} \sum_{i \in S} b_i(x_i)$$

$$\hat{X}(b_S^L) = \arg\max_{x \in X(L)} \sum_{i \in S} b_i(x_i)$$

Likewise, $w(b_{-S}^{-L})$ represents the value generated by bidders $N \setminus S$ and goods $K \setminus L$.

The core $\mathcal{C}(v)$ of a cooperative game consists of all feasible *payoffs* that are not blocked by any coalition (i.e., each group receives at least as much as it could achieve on its own so that it could not deviate and make all of its members better off). In our setting, since the seller owns all items, any group that does not include the seller cannot generate any value. Any coalition that contains the seller must receive at least what it could get from trading among itself. Therefore, the core consists of all payoff vectors $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ that satisfy the following constraints (where the first is feasibility and the rest assure no coalition can block the payoffs):

$$\begin{aligned} \pi_0 + \textstyle\sum_{i \in N} \pi_i &\leq w(v) \\ \pi_i &\geq 0 \quad \forall i \in N \\ \pi_0 + \textstyle\sum_{i \in S} \pi_i &\geq w(v_S) \quad \forall S \subseteq N \end{aligned}$$

In our setting with a single seller, the core is always nonempty. The payoffs $\pi_0 = w(v)$ and $\pi_i = 0$ for all $i \in N$ always satisfy these constraints.

A core-selecting auction is a direct mechanism (inducing a non-cooperative game among bidders) that maps bids on packages of goods to assignments and payments such that reported payoffs are in the core with respect to the bids. Denote bidder i's reported payoff as $\pi_i(b) =$

$b_i(x_i(b)) - p_i(b, x(b))$ and the seller's payoff as $\pi_0(b) = \sum_{i \in N} p_i(b, x(b))$. Then a core-selecting auction is defined as follows.

Definition 1. *A* core-selecting auction *is a direct mechanism that chooses $x(b)$ and $p(b, x(b))$ such that:*

$$\pi(b) = (\pi_0(b), \pi_1(b), \dots, \pi_n(b)) \in \mathcal{C}(b) \quad \forall b \in B$$

Many auctions fall into the class of core-selecting auctions. They all choose an optimal assignment $x(b) \in \hat{X}(b)$, which is necessitated by the first and last core inequalities. However, there may be infinitely many possible payments $p(b, x(b))$ that satisfy the core constraints for any given optimal assignment. Translating the constraints on payoffs into constraints on the assignment and payments yields:

$$\begin{aligned}
\pi_0(b) + \textstyle\sum_{i \in N} \pi_i(b) = w(b) \quad &\Leftrightarrow \quad x(b) \in \hat{X}(b) \qquad (1)\\
\pi_i(b) \geq 0 \quad \forall i \in N \quad &\Leftrightarrow \quad b_i(x_i(b)) \geq p_i(b, x(b)) \quad \forall i \in N\\
\pi_0(b) \geq 0 \quad &\Leftrightarrow \quad \textstyle\sum_{i \in N} p_i(b, x(b)) \geq 0\\
\textstyle\sum_{i \in S} \pi_i(b) \leq w(b) - w(b_{-S}) \quad \forall S \subset N \quad &\Leftrightarrow \quad \textstyle\sum_{i \in S} p_i(b, x(b)) \geq w(b_{-S}) - \sum_{i \in N \setminus S} b_i(x_i(b)) \quad \forall S \subset N
\end{aligned}$$

The second to last constraint is implied by the last constraints: $p_j(b, x(b)) \geq w(b_{-j}) - \sum_{i \neq j} b_i(x_i(b)) \geq 0$ prevents negative payments (otherwise the seller and the bidders in $N \setminus \{j\}$ could do better) and implies $\sum_{i \in N} p_i(b, x(b)) \geq 0$. Also, the second constraint implies that losing bidders pay zero $x_i(b) = \emptyset \implies p_i(b, x(b)) = 0$.

Some particular subclasses of core-selecting auctions that we consider are bidder-optimal core-selecting auctions, minimum-revenue core-selecting auctions, and pay-as-bid auctions. Bidder-optimal core-selecting auctions select payoffs on the bidder-Pareto-optimal frontier of the core.

Definition 2. *A* bidder-optimal core-selecting (BOCS) auction *is a core-selecting auction that chooses $x(b)$ and $p(b, x(b))$ such that, for all $b \in B$, there does not exist any $\hat{\pi} \in \mathcal{C}(b)$ such that $\hat{\pi}_i \geq \pi_i(b)$ for all $i \in N$ and the inequality is strict for at least one $i \in N$.*

There are generally many bidder-optimal payments, so a full specification of a particular BOCS auction requires a rule for choosing between them.[1] A subclass of BOCS auctions are those that minimize the seller's revenues.

Definition 3. *A* minimum-revenue core-selecting (MRCS) auction *is a core-selecting auction that chooses $x(b)$ and $p(b, x(b))$ such that, for all $b \in B$, the corresponding reported payoffs $\pi(b)$ solve:*

$$\min_{\pi \in \mathcal{C}(b)} \pi_0$$

Again, there are usually many MRCS payments because the seller's revenue can be split in different ways among the bidders.[2] On the opposite end of the spectrum are auctions that maximize the seller's revenues among reported core payoffs by having each bidder pay his full bid for his winning package.

Definition 4. *A* pay-as-bid (PAB) auction *is a core-selecting auction that chooses $x(b) \in \hat{X}(b)$ and $p_i(b, x(b)) = b_i(x_i(b))$.*

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[1] One example of a payment rule is to choose the bidder-optimal payment vector that is closest in Euclidean distance to the Vickrey payments as suggested by **?**.

[2] One can transfer the idea of the Vickrey-nearest payments by minimizing the Euclidean distance between the Vickrey payments and those payments that minimize revenue in the core. The resulting payments may differ from the bidder-optimal Vickrey-nearest payments. Other reference rules that minimize distances to reference points have been suggested by **?**.

Unlike in BOCS and MRCS auctions, in a PAB auction the payments are unique given a particular assignment. However, there are still multiple PAB auctions that differ in the way they choose among the optimal assignments when $\hat{X}(b)$ is not a singleton. Sometimes we will specify a *tie-breaking rule* as a way of describing a particular auction in this class (or a particular subset of auctions in other classes with more flexibility in choosing the payments).

We often make use of another kind of package auction that is not core-selecting: Vickrey or VCG auctions.

Definition 5. *A* Vickrey or VCG auction *chooses* $x(b) \in \hat{X}(b)$ *and* $p_i^V(b, x(b)) \equiv w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b))$ *for all* $i \in N$.

A Vickrey auction charges each participant the opportunity cost that his presence imposes on the other bidders, or equivalently, the minimum bid necessary for winning his assigned package. Notice that one of the core constraints requires that each bidder's payment be at least as large as his Vickrey payment $p_i^V(b)$, so we call this the *Vickrey constraint*. We also refer to Vickrey payoffs as $\pi_i^V(b) \equiv b_i(x_i(b)) - p_i^V(b, x(b)) = w(b) - w(b_{-i})$. Notice that the Vickrey payoff, in contrast to the Vickrey payment, does not depend on the assignment.

The highest reported payoff a bidder can receive in any core allocation is his Vickrey payoff. If $\pi_i > w(b) - w(b_{-i})$, then

$$\begin{aligned} \pi_0 + \sum_{j \neq i} \pi_j &= w(b) - \pi_i \\ &< w(b) - w(b) + w(b_{-i}) = w(b_{-i}), \end{aligned}$$

a contradiction to $\pi \in \mathcal{C}(b)$. The Vickrey payoff is always achievable in the core by any single bidder because the payoff profile $\pi_0 = w(b_{-i})$, $\pi_i = w(b) - w(b_{-i})$, and $\pi_j = 0$ for all $j \neq i$, always lies in the core. However, it may not be possible in a core allocation for all bidders to simultaneously receive their Vickrey payoffs. Thus, when the full Vickrey payoff vector $\pi^V(b) = (w(b) - \sum_{i \in N} \pi_i^V(b), \pi_1^V(b), \ldots, \pi_n^V(b))$ lies in the core, it is the unique bidder-optimal allocation and the unique outcome of any BOCS auction. When it is not in the core, there are multiple bidder-optimal allocations, all of which give the seller a larger payoff than π_0^V.[3] The bidder-pessimal allocation is always unique and is the outcome of the pay-as-bid auction: $(\pi_0(b), \pi_1(b), \ldots, \pi_n(b)) = (w(b), 0, \ldots, 0)$.

When the optimal assignment is not unique ($\hat{X}(b)$ contains more than one element), we will assume all auctions we consider apply the same *tie-breaking rule*: first give as many goods as possible to bidder 1, then give as many goods as possible to to bidder 2, etc. If some bidder i could receive two different packages of the same size, then favor goods with lower serial numbers. The specific choice of tie-breaking rule does not affect most of our results, but makes the exposition simpler.[4] This rule uniquely determines the assignment without randomization, so we write payments as a function only of bids: $p_i(b) \equiv p_i(b, x(b))$.

3. Connections Between Core-Selecting and Vickrey Auctions

When the Vickrey payoffs lie in the core, Vickrey and bidder-optimal core-selecting auctions produce identical outcomes. For the many settings in which the Vickrey payoffs are not in

[3] See Ausubel and Milgrom (2002), Theorem 6, and Ausubel and Milgrom (2006), Theorem 5.

[4] Aside from Lemma 1, the results will go through as long as all auction are following the same tie-breaking rule. Lemma 1 needs tie breaking in a way that does not change when the set of goods changes. However, if outcome additivity were defined over sets of tied assignments rather than the realized assignments, then the tie-breaking rule wouldn't matter and all Vickrey auctions would satisfy outcome additivity.

the core, the questions remains: how similar are these auctions? In this section, we will show how settings featuring unrelated goods shed light on the similarities between core-selecting and Vickrey auctions.

Definition 6. *Two sets of goods K and K' are* unrelated according to b *if they are disjoint and for all $y \subseteq K \cup K'$ and for all $i \in N$, $b_i(y) = b_i(y \cap K) + b_i(y \cap K')$.*

Unrelated goods, or unrelated sets of goods, have additive (reported) values for every bidder and thus do not exhibit any substitutability or complementarity. Given the additive nature of the unrelated goods, one might desire a mechanism that treats them additively. In other words, when unrelated goods are present according to some fixed reported values, the outcome of the mechanism equals the sum of the outcomes of that same mechanism applied separately to the unrelated sets. Notice that this is not a property about equilibria, but rather about the assignment and payments chosen by the mechanism given some fixed bids.

Definition 7. *A mechanism (x, p) satisfies* outcome additivity *if for any unrelated sets K and K' according to b, $x_i(b^{K\cup K'}) = x_i(b^K) \cup x_i(b^{K'})$ and $p_i(b^{K\cup K'}) = p_i(b^K) + p_i(b^{K'})$ for all $i \in N$.*

Lemma 1. *The Vickrey auction satisfies outcome additivity.*

Proof. If the optimal assignments are unique:

$$
\begin{aligned}
x^V(b^{K\cup K'}) &= \operatorname*{arg\,max}_{x\in X(K\cup K')} \sum_{i\in N} b_i(x_i) \\
&= \operatorname*{arg\,max}_{x\in X(K\cup K')} \sum_{i\in N} [b_i(x_i \cap K) + b_i(x_i \cap K')] \\
&= \left[\operatorname*{arg\,max}_{x^K\in X(K)} \sum_{i\in N} b_i(x_i^K)\right] \cup \left[\operatorname*{arg\,max}_{x^{K'}\in X(K')} \sum_{i\in N} b_i(x_i^{K'})\right] \\
&= x^V(b^K) \cup x^V(b^{K'}).
\end{aligned}
$$

The second equality follows from the definition of unrelated sets of goods and the third equality stems from the fact that $x \in X(K \cup K') \iff x \cap K \in X(K)$ and $x \cap K' \in X(K')$. If the tie-breaking rule determines the Vickrey assignment, then it must do so in the same way whether the unrelated sets are auctioned together or not. From the argument above, the set $\operatorname{arg\,max}_{x\in X(K\cup K')} \sum_{i\in N} b_i(x_i) = \operatorname{arg\,max}_{x^K\in X(K)} \sum_{i\in N} b_i(x_i^K) \times \operatorname{arg\,max}_{x^{K'}\in X(K')} \sum_{i\in N} b_i(x_i^{K'})$. The element that favors bidders with lower serial numbers from the left-hand side must be the sum of the elements that favor bidders with lower serial number from the right-hand side.

By the same argument, for all $S \subseteq N$,

$$
\begin{aligned}
w(b_S^{K\cup K'}) &= \max_{x\in X(K\cup K')} \sum_{i\in S} b_i(x_i) = \max_{x\in X(K\cup K')} \sum_{i\in S} [b_i(x_i \cap K) + b_i(x_i \cap K')] \\
&= \max_{x^K\in X(K)} \sum_{i\in S} b_i(x_i^K) + \max_{x^{K'}\in X(K')} \sum_{i\in S} b_i(x_i^{K'}) \\
&= w(b_S^K) + w(b_S^{K'}).
\end{aligned}
$$

This implies

$$\begin{aligned}
p_i^V(b^{K\cup K'}) &= b_i(x_i^V(b^{K\cup K'})) - [w(b^{K\cup K'}) - w(b_{-i}^{K\cup K'})] \\
&= b_i(x_i^V(b^K) \cup x_i^V(b^{K'})) - [w(b^K) + w(b^{K'})] \\
&\quad + [w(b_{-i}^K) + w(b_{-i}^{K'})] \\
&= b_i(x_i^V(b^K)) + b_i(x_i^V(b^{K'})) - w(b^K) - w(b^{K'}) \\
&\quad + w(b_{-i}^K) + w(b_{-i}^{K'}) \\
&= p_i^V(b^K) + p_i^V(b^{K'}).
\end{aligned}$$

□

Core-selecting auctions do not share the outcome additivity of the Vickrey auction in the presence of unrelated goods (see Corollary 1). Intuitively, combining unrelated goods into one auction makes it harder for any coalition to block a payoff vector. The seller has more to lose from abandoning the auction for an alternative deal with some subset of the bidders because he must give up the revenue from a larger set of items sold to the other bidders. The next theorem exploits this fact. It shows that you can take any setup of bidders, items, and valuations and make all bidder-optimal core-selecting auctions result in the same bidder payoffs as the Vickrey auction, simply by adding some unrelated items to the sale. The inclusion of these unrelated items does not change the bidders' Vickrey payoffs. Rather, it relaxes the other core constraints so that the original Vickrey payoffs become part of the new core.

Theorem 1. *Take any set of bidders N, any set of items K, and any bids $b_i : 2^K \to \mathbb{R}_+$ for $i \in N$. Then there exist additional items $U = \{u_j\}_{j\in J}$ and extended bids $\tilde{b}_i : 2^{K\cup U} \to \mathbb{R}_+$ such that:*

(i) $\tilde{b}^K = b$

(ii) $K, u_1, \ldots, u_r$ *are pairwise unrelated according to* $\tilde{b}$

(iii) $\pi^V(\tilde{b}) \in \mathcal{C}(\tilde{b})$

(iv) $\pi_i^V(\tilde{b}) = \pi_i^V(b) \quad \forall i \in N$

Proof. The Vickrey payoffs $\pi_i(\tilde{b})$ are in $\mathcal{C}(\tilde{b})$ if and only if, for all $S \subseteq N$,

$$\sum_{i\in S}[w(\tilde{b}) - w(\tilde{b}_{-i})] \le w(\tilde{b}) - w(\tilde{b}_{-S}). \tag{2}$$

Start with $\tilde{b} \equiv b$. If (2) indeed holds for all $S \subseteq N$, then $U = \emptyset$ and we are done. If not, then begin with any $S \subseteq N$ for which (2) is violated. Add item u_1 to U and define

$$\alpha_1 = \sum_{i\in S}[w(\tilde{b}^K) - w(\tilde{b}_{-i}^K)] - [w(\tilde{b}^K) - w(\tilde{b}_{-S}^K)] \tag{3}$$

and

$$\tilde{b}_i(u_1) = \begin{cases} \alpha_1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases} \tag{4}$$

Let $\tilde{b}_i(y) \equiv b_i(y \cap K) + \sum_{j:u_j\in y} \tilde{b}_i(u_j)$ for all $i \in N$ and for all $y \in 2^{K\cup U}$. Note that (i) and (ii) hold by construction.

Since every bidder in S has the same value for u_1, any group S' such that $S' \cap S \neq \emptyset$ has value $w(\tilde{b}_{S'}) = w(\tilde{b}_{S'}^K) + \alpha_1$. This equation holds for $S' = N$ and for all sets S' of size $n-1$ because

$|S| > 1$. (Inequality (2) holds trivially for sets consisting of only a single bidder.) Therefore, no bidder's marginal contribution to the grand coalition (i.e., Vickrey payoff) has changed, so $\pi_i^V(\tilde{b}) = \pi_i^V(b)\ \forall i \in N$.

Also, because all bidders that are not in S have zero value for the new good, $w(\tilde{b}_{-S}) = w(\tilde{b}^K_{-S})$. Thus,

$$\begin{aligned}\sum_{i\in S}[w(\tilde{b}) - w(\tilde{b}_{-i})] &= \sum_{i\in S}[(w(\tilde{b}^K) + \alpha_1) - (w(\tilde{b}^K_{-i}) + \alpha_1)] \\ &= \sum_{i\in S}[w(\tilde{b}^K) - w(\tilde{b}^K_{-i})] \\ &= [w(\tilde{b}^K) - w(\tilde{b}^K_{-S})] + \alpha_1 \\ &= w(\tilde{b}) - w(\tilde{b}_{-S}).\end{aligned}$$

The inequality (2) is now satisfied for the chosen S.

Next, notice that for any $S' \neq S$, the left-hand side of (2) has not changed while the right-hand side has become weakly larger. This is because $w(\tilde{b}) = w(\tilde{b}^K) + \alpha_1$ and $w(\tilde{b}_{-S'}) \in \{w(\tilde{b}^K_{-S'}) + \alpha_1, w(\tilde{b}^K_{-S'})\}$. Therefore, if (2) held for S' with K, then it still holds with $K \cup U$.

Repeat this process until (2) is satisfied for all $S \subseteq N$. □

Given some fixed bids, the bidders' reported Vickrey payoffs can always be part of the core with respect to bids, at least in some extended setting. Moreover, this extended setting has not changed anything fundamental about the original set of goods or their reported values because they are unrelated to the additional items. Since these additional goods are unrelated and valued by at least two bidders, all of the Vickrey payoff still comes from the original set of goods. So, in this sense, the Vickrey auction is very similar to any bidder-optimal core-selecting auction, at least as it applies to a fixed set of bids. The following theorem shows that any Nash equilibrium of a Vickrey auction is part of a Nash equilibrium of any bidder-optimal core-selecting auction over the (possibly) extended set of goods from Theorem 1. So even the equilibrium payoffs will be similar.

Theorem 2. *Take any set of bidders N, goods K, and values v. Suppose b^* is a Nash equilibrium of the Vickrey auction. Then there exists an extended set of goods $K \cup U$ and extended true values $\tilde{v}(y) \equiv v(y \cap K) + v'(y \cap U)$ for all $y \subseteq K \cup U$ such that $\tilde{b}^*(y) = b^*(y \cap K) + v'(y \cap U)$ is a Nash equilibrium of every bidder-optimal core-selecting auction, yielding equilibrium assignment $x_i^V(b^*) \cup x_i^V(v')$ for all $i \in N$, equilibrium payments $p_i^V(b^*) + p_i^V(v')$ for all $i \in N$, and equilibrium bidder payoffs $\pi_i^V(b^*)$ for all $i \in N$.*

Proof. Construct U and $\tilde{v}$ as in Theorem 1. Then the result follows from two steps.

1. We will prove that $\tilde{b}^*(y) = b^*(y \cap K) + v'(y \cap U)$ is a Nash equilibrium of the Vickrey auction when the setting is $(N, K \cup U, \tilde{v})$. First note that v' is an equilibrium of the Vickrey auction in the setting (N, U, v') because truthful bidding is always an equilibrium of the Vickrey

auction. Then, for every bidder i,

$$\begin{aligned}\tilde{v}_i(x_i^V(\tilde{b}^*)) - p_i^V(\tilde{b}^*) &= \tilde{v}_i(x_i^V(b^*) \cup x_i^V(v')) \\ &\quad - [p_i^V(b^*) + p_i^V(v')] \\ &= v_i(x_i^V(b^*)) + v_i'(x_i^V(v')) \\ &\quad - p_i^V(b^*) - p_i^V(v') \\ &= \left[v_i(x_i^V(b^*_{-i}, v_i)) - p_i^V(b^*_{-i}, v_i)\right] \\ &\quad + \left[v_i'(x_i^V(v')) - p_i^V(v')\right] \\ &= \tilde{v}_i(x_i^V(\tilde{b}^*_{-i}, \tilde{v}_i)) - p_i^V(\tilde{b}^*_{-i}, \tilde{v}_i)\end{aligned}$$

The first equality is because the Vickrey auction satisfies outcome additivity. The second equality is because K is unrelated to U. The third equality is because b^* must give bidder i the same payoff as (b^*_{-i}, v_i). (Truthful bidding is always a best response in the Vickrey auction, so any strategy played in a Nash equilibrium must offer the same payoff.) Together, these steps show that $\tilde{b}^*_i$ is a best response to $\tilde{b}^*_{-i}$ for all i because it yields the same payoff as truthful bidding.

2. We will prove that whenever the payoff allocation associated with an equilibrium of the Vickrey auction $\tilde{b}^*$ is in $\mathcal{C}(\tilde{b}^*)$, then $\tilde{b}^*$ is also an equilibrium of every bidder-optimal core-selecting auction. If $\pi^V(\tilde{b}^*) \in \mathcal{C}(\tilde{b}^*)$, every bidder-optimal core-selecting auction chooses the same assignment and payments as the Vickrey auction, which are $x_i^V(\tilde{b}^*) = x_i^V(b^*) \cup x_i^V(v')$ and $p_i^V(\tilde{b}^*) = p_i^V(b^*) + p_i^V(v')$ for all $i \in N$ because of the outcome additivity of the Vickrey auction. First, because truthful bidding is a weakly dominant strategy in the Vickrey auction, our considered equilibrium $\tilde{b}^*$ must give bidder i the same payoff as $(\tilde{v}_i, \tilde{b}^*_{-i})$. Given the bids of the other players, the best possible payoff for bidder i in any core-selecting auction occurs when he bids truthfully and receives his Vickrey payoff: $\pi_i^V(\tilde{v}_i, \tilde{b}^*_{-i})$. Since $\pi^V(\tilde{b}^*) \in \mathcal{C}(\tilde{b}^*)$ by construction, any bidder-optimal core-selecting auction must give bidder i a payoff $\tilde{v}_i(x_i^V(\tilde{b}^*)) - p_i^V(\tilde{b}^*)$, which equals $\pi_i^V(\tilde{v}_i, \tilde{b}^*_{-i})$ by step 1. Thus, $\tilde{b}^*_i$ is a best response to $\tilde{b}^*_{-i}$ for all $i \in N$ and $\tilde{b}^*$ is also an equilibrium of every bidder-optimal core-selecting auction.

□

Thus, any equilibrium outcome of the Vickrey auction in *any* setting, even those in which the Vickrey payoffs are not in the core, is a part of an equilibrium outcome of any bidder-optimal core-selecting auction in a possibly extended setting. This similarity means that many of the flaws of the Vickrey auction carry over into all bidder-optimal core-selecting auctions. When the Vickrey auction produces low revenue, the component of the revenue that stems from the original set of goods in any bidder-optimal core-selecting auction over the extended setting will be low as well. When the revenue from the Vickrey auction is non-monotonic in bids, the revenue in any bidder-optimal core-selecting auction in an extended setting will be non-monotonic as well. Non-monotonic revenue gives incentives for collusion that will be present in every bidder-optimal core-selecting auction.

Note that when we talk about non-monotonic revenue, we need to compare two different bid profiles b. Thus, to ensure the reported Vickrey payoffs are in the reported core under both bid profiles, the extended setting must incorporate enough unrelated goods to cover both situations.

Theorem 3. *Consider two bid profiles b and b' over the set of goods K such that $b < b'$ and $\pi_0^V(b) > \pi_0^V(b')$. There exist additional items $U = \{u_j\}_{j \in J}$ and extended bids $\tilde{b}_i : 2^{K \cup U} \to \mathbb{R}_+$ and $\tilde{b}'_i : 2^{K \cup U} \to \mathbb{R}_+$ such that $\tilde{b} < \tilde{b}'$ and $\pi_0^{BOCS}(\tilde{b}) > \pi_0^{BOCS}(\tilde{b}')$ for all bidder-optimal core-selecting auctions.*

Proof. First construct U_b and $\tilde{b}$ for b and $U_{b'}$ for $\tilde{b}'$ for b' as in Theorem 1. Then define $U \equiv U_{\tilde{b}} \cup U_{\tilde{b}'}$, $\tilde{b}_i(y) \equiv \tilde{b}'_i(y)$ for all $y \in U_{b'}$ and all $i \in N$, and $\tilde{b}'_i(y) \equiv \tilde{b}_i(y)$ for all $y \in U_b$ and $i \in N$. Recursively define $\tilde{b}_i(y) = b(y \cap K) + \tilde{b}_i(y \cap U)$ for all $y \in K \cup U$ and all $i \in N$ and $\tilde{b}'_i(y) = b(y \cap K) + \tilde{b}'_i(y \cap U)$ for all $y \in K \cup U$ and all $i \in N$.

By construction, $\tilde{b} < \tilde{b}'$ because $\tilde{b}^U = \tilde{b}'^U$ and $\tilde{b}^K = b < b' = \tilde{b}'^K$. Having more unrelated goods than necessary cannot change the results of Theorem 1, as noted in the proof. Therefore, by construction and Theorem 1, $\pi^V(\tilde{b}) \in \mathcal{C}(\tilde{b})$ and $\pi^V(\tilde{b}') \in \mathcal{C}(\tilde{b}')$. This implies

$$\begin{aligned}
\pi_0^{BOCS}(\tilde{b}) = \pi_0^V(\tilde{b}) &= \sum_{i \in N} p_i^V(\tilde{b}) \\
&= \sum_{i \in N} p_i^V(\tilde{b}^K) + p_i^V(\tilde{b}^U) \\
&= \sum_{i \in N} p_i^V(b) + p_i^V(\tilde{b}'^U) \\
&> \sum_{i \in N} p_i^V(b') + p_i^V(\tilde{b}'^U) \\
&= \sum_{i \in N} p_i^V(\tilde{b}'^K) + p_i^V(\tilde{b}'^U) \\
&= \sum_{i \in N} p_i^V(\tilde{b}') = \pi_0^V(\tilde{b}') = \pi_0^{BOCS}(\tilde{b}')
\end{aligned}$$

□

For every pair of bid profiles for which the Vickrey auction exhibits non-monotonic revenues, there exists a (potentially) different pair of bid profiles for which every bidder-optimal core-selecting auction exhibits non-monotonic revenues. Note that the (potentially) different pair of bid profiles is a pair for which both bidder-optimal core-selecting auctions and the Vickrey auction exhibit non-monotonic revenues. However, this does not imply that set of pairs of bid profiles for which bidder-optimal core-selecting auctions exhibit non-monotonic revenues is a subset of that of the Vickrey auction. There exist pairs of bid profiles for which bidder-optimal core-selecting auctions exhibit non-monotonic revenues and the Vickrey auction does not.

Consider the following pair of bid profiles for goods $K = \{A, B, C, D\}$. Assume bids for all packages not shown in the table are zero. Also note that we calculate reported payoffs for minimum-revenue core-selecting auctions, which are a subset of bidder-optimal core-selecting auctions. They all lead to a unique reported payoff vector that differs from that of the Vickrey auction and exhibits non-monotonic revenues while the Vickrey auction does not.

	A	B	C	D	AD	BD	CD	ABC		
									$\pi_0^V = 8$	$\pi_0^{\text{MRCS}} = 13$
b_1	4	0	0	8	12	8	8	5	$\pi_1^V = 4$	$\pi_1^{\text{MRCS}} = 3$
b_2	0	5	0	8	8	13	8	5	$\pi_2^V = 5$	$\pi_2^{\text{MRCS}} = 3$
b_3	0	0	5	8	8	8	13	5	$\pi_3^V = 5$	$\pi_3^{\text{MRCS}} = 3$
b_4	0	0	0	0	0	0	0	8	$\pi_4^V = 0$	$\pi_4^{\text{MRCS}} = 0$

The bids b report that $\{A, B, C\}$ and $\{D\}$ are unrelated. However, in this case $\pi^V(b) \notin \mathcal{C}(b)$, meaning that more unrelated goods would need to be added to complete the process in Theorem 1. This is done purposely so that the seller's revenues can change differently when bidder 1's bids for A, AD, and ABC are increased.

	A	B	C	D	AD	BD	CD	ABC		
b'_1	5	0	0	8	13	8	8	5	$\pi_0^V = 8$	$\pi_0^{\text{MRCS}} = 12.5$
b'_2	0	5	0	8	8	13	8	5	$\pi_1^V = 5$	$\pi_1^{\text{MRCS}} = 3.5$
b'_3	0	0	5	8	8	8	13	5	$\pi_2^V = 5$	$\pi_2^{\text{MRCS}} = 3.5$
b'_4	0	0	0	0	0	0	0	8	$\pi_3^V = 5$	$\pi_3^{\text{MRCS}} = 3.5$
									$\pi_4^V = 0$	$\pi_4^{\text{MRCS}} = 0$

The bids b' relax the binding core constraints $\pi_i + \pi_j \leq w(b) - w(b_{-\{i,j\}})$ for $i \neq j \in \{1, 2, 3\}$, allowing higher payoffs for the bidders and lower revenues for the seller. However, they do not change the Vickrey constraints and so do not influence the seller's revenues in the Vickrey auction.

In the previous example, one unrelated good was not enough to equate Vickrey and bidder-optimal core-selecting payoffs. Generally, how many unrelated goods must we include to make the reported Vickrey payoffs in the reported core? To answer this question, we must define the number of contributing bidders: those that make a non-zero marginal contribution to the grand coalition. Note that a bidder must win $x_i(b) \neq \emptyset$ to contribute, but the converse is not true. So the contributing bidders are a subset of the winning bidders.

Definition 8. *The number of contributing bidders is* $m(b) = |\{i \in N | w(b) - w(b_{-i}) > 0\}|$.

Theorem 4. *Take any set of bidders N, any set of items K, and any bids $b_i : 2^K \to \mathbb{R}_+$ for $i \in N$. There exists a set of additional items U that satisfies Theorem 1 such that $|U| \leq \binom{m(b)}{2}$. For some b, this upper bound on $|U|$ is attained.*

Proof. Denote the set of contributing bidders in the original setup as $Z(b)$ or Z. Take any $S \subseteq N$. The left-hand side of (2) is the same for S and $S \cap Z$ because non-contributing bidders have reported Vickrey payoffs of zero. The right-hand side is smaller for $S \cap Z$ than for S because $w(b_{-S}) \leq w(b_{-S \cap Z})$. Then (2) is satisfied for $S \subseteq N$ if it is satisfied for $S \cap Z$ and we may ignore all constraints that include non-contributing bidders.

Suppose (2) is violated for every $S \subseteq Z$ such that $|S| = 2$. (Remember that $|S| \geq 2$ for any violated constraint.) There are $\binom{m(b)}{2}$ such sets. For each such S, add item u_S to U. Define

$$\alpha_S = \max_{S \subseteq S' \subseteq Z} \sum_{i \in S'} [w(b) - w(b_{-i})] - [w(b) - w(b_{-S'})]$$

and

$$\tilde{b}_i(u_S) = \begin{cases} \alpha_S & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Now (2) is satisfied for every $S' \subseteq Z$ because adding u_S with value as indicated raises the right-hand side of (2) for each $S' \supseteq S$ by α_S and does not change the left-hand side.

Next we must show that for each $m \in \mathbb{N}_+$, there exists a setting (N, K, b) such that $m(b) = m$ and the smallest U that satisfies Theorem 1 has size $|U| = \binom{m(b)}{2}$. If $m(b) = 0$, every bidder has a Vickrey payoff equal to zero and, therefore, the Vickrey payoff lies in $\mathcal{C}(b)$. Thus, $U = \emptyset$ and $|U| = 0$. If $m(b) = 1$, then $\pi^V(b) \in \mathcal{C}(b)$ because (2) is satisfied for all subsets $S \subset N$ of size 1. Thus, $U = \emptyset$ and $|U| = 0$.

Take any $m > 1$. Let $N = \{1, ..., m+1\}$ and $K = \{k_{j,l} : j \neq l \in N \setminus \{m+1\}\}$ so that $|K| = m^2 - m$. For all $i \leq m$ and for all $j \neq l \in N \setminus \{m+1\}$, define

$$b_i(k_{j,l}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and $b_i(y) = \sum_{k_{j,l} \in y} b_i(k_{j,l})$ for all $y \subseteq K$. For bidder $m+1$, define $b_{m+1}(k_{j,l}) = 0$ for all $j \neq l \in N \setminus \{m+1\}$ and $b_{m+1}(\{k_{j,l}, k_{l,j}\}) = 1$. Let bidder $m+1$ have additive preferences for all larger packages.

Each bidder $i \leq m$ wins $x_i(b) = \{k_{j,l} \in K : j = i\}$ and has a marginal contribution to the grand coalition equal to $w(b) - w(b_{-i}) = (m^2 - m) - (m^2 - 2m + 1) = m - 1 = b_i(x_i(b))$. Bidder $m+1$ loses and has a marginal contribution of zero. So $m(b) = m$. However, $w(b) - w(b_{-\{i,j\}}) = (m^2 - m) - (m^2 - 3m + 3) = 2m - 3$. Thus (2) is violated for each pair $j \neq l \in N \setminus \{m+1\}$. To keep the Vickrey payoffs the same, at least two bidders must have value for each additional, unrelated good. To fix (2) for any $S = \{i, j\}$, we must raise $w(b)$ without raising $w(b_{-\{i,j\}})$. Therefore, the unrelated good cannot be valued by any bidder in $N \setminus \{i, j\}$. So there are only two candidates for bidders to value the unrelated good, i and j. And since we need at least two bidders to keep the Vickrey payoffs the same, we must add a good valued by just i and j. This good will not fix any of the other violated constraints because either i or j appears in both terms on the right-hand side. Therefore, we need a different good for each of the $\binom{m}{2}$ violated constraints and the minimum size of U is $\binom{m}{2}$. □

It may take at most $\binom{m(b)}{2} = \frac{m(b)(m(b)-1)}{2}$ unrelated goods to force the Vickrey payoffs from an arbitrary setting into the core. This upper bound grows faster than the number of contributing bidders because the number of relevant core constraints, and hence the number of constraints that may need to be relaxed, expands faster than the number of contributing bidders. However, relatively simple settings already expose the Vickrey auction's shortcomings. The following example demonstrates how a single unrelated good can link bidder-optimal and Vickrey outcomes.

Example 1: Complementary Goods

	A	B	AB
b_1	10	0	10
b_2	0	10	10
b_3	0	0	10

$\pi_0^V = 0$
$\pi_1^V = 10$
$\pi_2^V = 10$
$\pi_3^V = 0$

The table shows bids for a set of goods $\{A, B\}$. This type of example with two goods that are complements for one of three bidders has been used to convey both the weaknesses of the Vickrey auction and the solution provided by core-selecting auctions. The seller's revenue is zero in the Vickrey auction even though bidder 3 is willing to pay 10. If bidder 1's bids for A and AB drop by 1, the seller's revenue actually increases to 1. This non-monotonicity means that if bidders 1 and 2 are better off colluding and both bidding 10.

All bidder-optimal core-selecting auctions avoid these issues by guaranteeing the seller 10 in revenue, even when bidder 1's bid decreases.[5]

These differences disappear when the auction incorporates one unrelated good, C.

Example 2: Complementary and Unrelated Goods

	A	B	C	AB	AC	BC	ABC
b_1	10	0	10	10	20	10	20
b_2	0	10	10	10	10	20	20
b_3	0	0	0	10	0	0	10

$\pi_0^V = 10$
$\pi_1^V = 10$
$\pi_2^V = 10$
$\pi_3^V = 0$

[5] Of course, these bids cannot be a full-information Nash equilibrium of any bidder-optimal core-selecting auction since the payment made by either bidder 1 or bidder 2 must be higher than his Vickrey payment.

Notice that all values for C are additive, so it is indeed an unrelated item. The Vickrey payoffs for bidders 1 and 2 are still 10 because whoever wins C pays a full 10 for it. The revenue generated by goods A and B in the Vickrey auction is still zero and a reduction in bidder 1's bids for all packages containing good A is accompanied by a corresponding increase in revenue.

The Vickrey payoffs now lie in the core. The seller can no longer threaten to block these payoffs by selling AB to bidder 3 because to do this he would have to give up the 10 he makes on item C. Thus, the seller's revenue exhibits non-monotonicity and this provides incentives for collusion.

The seller actually receives as much revenue, 10, as he did in any bidder-optimal core-selecting auction without the unrelated item. Though this revenue meets the revenue standard set by the core, it seems low for a different reason – because he receives no additional revenue from selling A and B. The next section explores the sense in which this revenue is low by identifying a criterion for revenue that is not satisfied by the bidder-optimal core-selecting auctions in this setting with the unrelated item.

4. Low Revenue in Core-Selecting Auctions

When an auction does not satisfy outcome additivity, increasing the number of goods can reduce competition and generate less revenues than auctioning them separately (using the same auction rules). In the previous example, the seller receives revenues of 10. This is enough money to deter any possible blocking coalition and create a stable outcome. However, it is less revenue than the seller would receive if he ran two separate bidder-optimal core-selecting auctions – one for $\{A, B\}$ and one for C. With these separate auctions, he would make 10 in each auction, for a total revenue of 20. The next theorem generalizes this principle.

Theorem 5. *Given reported value functions b_i for $i \in N$, suppose there exists a partition $\{K_1, ..., K_r\}$ of the set of items K such that K_j and K_l are unrelated for all $j \neq l \in \{1, \ldots, r\}$. Pick any minimum-revenue core-selecting auction. Running this auction separately for each K_j yields at least as much revenue, and for some bid profiles strictly greater revenue, as running this auction once for the entire set K.*

Proof. Given any set of goods $L \subseteq K$, the core constraints are:

$$\sum_{i \in S} \pi_i \leq w(b^L) - w(b^L_{-S}) \quad \forall S \subseteq N \tag{5}$$

Any minimum-revenue core-selecting auction maximizes the sum of bidder payoffs $\sum_{i \in N} \pi_i$ subject to these constraints. For any $S \subseteq N$,

$$\begin{aligned} w(b^K_S) &= \max_{x \in X(K)} \sum_{i \in S} b_i(x_i) = \max_{x \in X(K)} \sum_{i \in S} \sum_{j=1}^{r} b_i(x_i \cap K_j) \\ &= \sum_{j=1}^{r} \max_{x \in X(K_j)} \sum_{i \in S} b_i(x_i) = \sum_{j=1}^{r} w(b^{K_j}_S). \end{aligned}$$

Then, by (5), for any minimum-revenue allocations $\pi^j \in \mathcal{C}(b^{K_j})$ and for all $S \subseteq N$,

$$\sum_{j=1}^{r} \sum_{i \in S} \pi^j_i \leq \sum_{j=1}^{r} [w(b^{K_j}) - w(b^{K_j}_{-S})] = w(b^K) - w(b_{-S}).$$

Thus, $\pi = \sum_{j=1}^{r} \pi^j \in \mathcal{C}(b^K)$ and the seller's revenue $\pi_0 = \sum_{j=1}^{r} \pi^j_0$ is at least as large as his minimum revenue over all vectors in $\mathcal{C}(b^K)$.

Next, we will use an example to show that the revenue can be strictly greater with multiple minimum-revenue auctions rather than with one large minimum-revenue auction. Consider the bid profile over the set $K = \{A, B, C\}$ from Example 2.

	A	B	C	AB	AC	BC	ABC
b_1	10	0	10	10	20	10	20
b_2	0	10	10	10	10	20	20
b_3	0	0	0	10	0	0	10

When the entire set K is auctioned together, the Vickrey payoffs $\pi_0^V = \pi_1^V = \pi_2^V = 10, \pi_3^V = 0$ are in the core, so the Vickrey payoff profile is the unique minimum-revenue allocation and must be chosen by any minimum-revenue core-selecting auction. The seller receives revenue of 10. Now partition K into $K_1 = \{A, B\}$ and $K_2 = \{C\}$. $\mathcal{C}(b^{K_2})$ consists of a single payoff vector: $\pi_0^2 = 10$, $\pi_1^2 = \pi_2^2 = \pi_3^2 = 0$. In any minimum-revenue allocation in $\mathcal{C}(b^{K_1})$, the seller receives revenue of $\pi_0^1 = 10$. Therefore, the seller receives a total revenue of 20 from the two auctions, which is strictly greater than his revenue from the single auction involving all of the goods in K. □

Corollary 1. *Bidder-optimal core-selecting auctions do not satisfy outcome additivity.*

Proof. In example 2, used in the proof of Theorem 5, all bidder-optimal (and, therefore, also minimum-revenue) core-selecting auctions choose the same assignment and payments and the payments do not satisfy outcome additivity. □

Notice that we ignored strategic considerations in Theorem 5 by fixing reported values. Core-selecting auctions do not share the dominant-strategy incentive compatibility of the Vickrey auction, so bidders may not report their values truthfully. However, Day and Milgrom (2008) proved that there exist Nash equilibria of every core-selecting auction in which the assignment is optimal and the payoffs are bidder-optimal in the core with respect to the true values. So our results imply equilibrium revenue can be higher with separate minimum-revenue core-selecting auctions than with one big minimum-revenue core-selecting auction.

This sensitivity of bidder payments to the grouping of items for auction is a major difference between minimum-revenue core-selecting auctions and the Vickrey auction, which has additive payments by Lemma 1. Moreover, when goods are truly unrelated, separating them into different auctions does not hurt the efficiency of the assignment. So minimum-revenue core-selecting auctions are not immune to manipulation by the seller in the way he packages his items for sale. They don't guarantee the seller a high enough revenue to prevent him from wanting to withdraw a subset of his items and save them for a later auction.

What is the correct revenue standard? The answer depends on the seller's ability to design the auction. If he is not the designer but merely a player who can participate or not, then the core is good standard. However, even in this case, Theorem 5 highlights an interesting property of minimum-revenue core-selecting auctions. Namely, they create incentives for the seller to create false identities and show up as multiple sellers. By splitting ownership of the goods, he can achieve the revenue from auctioning the goods separately. This incentive does not exist in the Vickrey auction because it satisfies outcome additivity. Thus, core-selecting auctions can increase incentives for the sellers to create false identities while eliminating those incentives for bidders.

The tradeoff between bidders and sellers wanting to create false identities can be remedied with other (not minimum-revenue) core-selecting auctions. The revenue properties of other (not minimum-revenue) core-selecting auctions can mimic the additivity of the Vickrey auction.

Theorem 6. *The pay-as-bid auction satisfies outcome additivity.*

Proof. The pay-as-bid auction chooses assignment $x(b^K) = x^V(b^K)$, which satisfies additivity by Lemma 1. The pay-as-bid auction chooses payments $p_i(b^K) = b_i(x_i^V(b^K))$, which trivially satisfies additivity because it is satisfied by $x_i^V(b^K)$. □

Theorem 7. *There exist core-selecting auctions other than the pay-as-bid auction that satisfy outcome additivity.*

Proof. Any core-selecting auction assigns the goods in the same way as the Vickrey auction and, thus, has an additive assignment function by the proof of Lemma 1. So we need only to find a core-selecting auction with an additive payment function.

Pick any minimum-revenue core-selecting auction with payment rule $\tilde{p}$. Now consider the following auction mechanism (x^a, p^a):

1. Assign goods according to $x^a(b^K) \equiv x^V(b^K)$ (the optimal assignment).
2. Partition K into sets $\{K_1, ..., K_r\}$ such that r is as large as possible and K_j and K_l are unrelated according to b for every $j \neq l \in \{1, \ldots, r\}$.
3. Determine payments by the rule $p^a(b^K) \equiv \sum_{j=1}^r \tilde{p}(b^{K_j})$.

Since the auction has an additive assignment function, $\pi_i^a(b^K) \equiv b_i(x_i^a(b^K)) - p_i^a(b^K) = b_i(x_i^V(b^K)) - \sum_{j=1}^r \tilde{p}_i(b^{K_j}) = \sum_{j=1}^r b_i(x_i^V(b^{K_j})) - \sum_{j=1}^r \tilde{p}_i(b^{K_j}) = \sum_{j=1}^r \tilde{\pi}_i(b^{K_j})$ where $\tilde{\pi}_i(b^{K_j})$ is the payoff associated with the minimum-revenue core-selecting auction $(x^V, \tilde{p})$. By the proof of Theorem 5, $\pi^a(b^K) = \sum_{j=1}^r \tilde{\pi}(b^{K_j}) \in \mathcal{C}(b^K)$ so (x^a, p^a) as described is a core-selecting auction.

Take any two unrelated sets K and K'. Let $\{K_1, ..., K_r\}$ and $\{K'_1, ..., K'_t\}$ be the sets from part 2 for bid profiles (b^K) and $(b^{K'})$, respectively. Then, for bids $b^{K \cup K'}$, the second step of the auction yields the partition $\{K_1, ..., K_r, K'_1, ..., K'_t\}$. Therefore, $p^a(b^{K \cup K'}) = \sum_{j=1}^r \tilde{p}(b^{K_j}) + \sum_{j=1}^t \tilde{p}(b^{K'_j}) = p^a(b^K) + p^a(b^{K'})$. □

Given a fixed set of inputs, these core-selecting auctions that satisfy outcome additivity guarantee that the seller's revenue is additive with respect to unrelated sets of goods, thereby eliminating any incentive for him to auction the unrelated goods separately or disaggregate into false identities. Of course, that means they also offer the seller a (sometimes strictly) higher amount of revenue than minimum-revenue core-selecting auctions.

However, the auctions in the proof of Theorem 7 also give the bidders the incentive to misrepresent their preferences to reduce their payments. They will want to report strict (but small) complementarity or substitutability to make the partition in part 2 a singleton and to reduce the payments to $\tilde{p}(b^K)$. Then in equilibrium, the seller will not end up receiving higher revenue using this core-selecting auction that satisfies outcome additivity than he would have gotten by using a minimum-revenue core-selecting auction.

If the seller knows how to separate his set of goods into groups which are unrelated for the bidders according to their true valuations, then he could run separate minimum-revenue core-selecting auctions without distorting the bidders' incentives. Or, he could run a modified version of auction (x^a, p^a) in Theorem 7 in which he determines the partition in part 2 based on his prior knowledge of unrelated goods, rather than the reported values. This auction will have complete-information Nash equilibria whose assignment is optimal and whose payoffs are in the core with respect to the true values. Thus, even if the seller does not know much about the bidders' valuations, he might improve his revenue if he can identify unrelated goods.

5. Conclusion

Minimum-revenue core-selecting auctions suffer many of the same problems as the Vickrey auction because, by adding unrelated goods, we can embed any setting into one in which the

Vickrey payoffs are in the core. This transformation requires no more than $\binom{m(b)}{2}$ extra, unrelated goods. Moreover, these results hold not only for fixed bid profiles, but also in Nash equilibrium. This means that many properties of the Vickrey auction carry over into all bidder-optimal core-selecting auctions. One property not shared by these two types of auction is outcome additivity. The Vickrey auction satisfies outcome additivity but minimum-revenue core-selecting auctions do not. The latter do not have additive payment rules. Therefore, revenue is higher in minimum-revenue core-selecting auctions when unrelated goods are auctioned separately. There exist many other (not minimum-revenue) core-selecting auctions that satisfy outcome additivity, but provide stronger incentives for bidders to misrepresent their preferences.

All of these conclusions rely on the presence of unrelated goods, so one might consider how often such values actually occur, especially in the specific form used in Theorem 1. There are examples of real auctions combining seemingly unrelated goods, such as some European spectrum auctions that combine bidding for low and high frequency spectrum. But all of our results would hold for goods that are mostly unrelated, in other words, goods that have a small amount of substitutability or complementarity. What does small mean? Small enough to not affect the optimal assignment or payments. This type of relationship between values is much harder to recognize and does not allow the remedies proposed in Section 4. Auctioning these items separately will not permit bidders to fully express their true preferences. Even if one big core-selecting auction like (x^a, p^a) determined the assignment and then calculated payments as if the mostly unrelated goods had been auction separately, it would be difficult to determine how to divide the goods.

Despite the perverse revenue incentives for sellers to unbundle their items in minimum-revenue core-selecting auctions that were designed for combining goods into one sale, those auctions share many properties – good and bad – of the Vickrey package auction. Unrelated goods help demonstrate the many similarities between the two types of auctions as well as the relationship between the number of goods being auctioned and the stability of the Vickrey outcome against renegotiation.

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