Nash Equilibria of Sealed-Bid Combinatorial Auctions*

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Abstract

The complex bidding space of combinatorial auctions has led the literature to focus on specific full-information Nash equilibria. This analysis broadens the view to the complete set of full-information, pure-strategy Nash equilibria of all prevalent sealed-bid combinatorial auctions. Using two intuitive conditions on the assignment and prices induced by the equilibrium bids, we characterize every Nash equilibrium of auctions that assign bundles to maximize the sum of bids and choose bidder payments between the minimum bid required to win a bundle and the bid amount. These conditions show that the equilibria of the pay-as-bid (PAB) auction are a subset of the equilibria of bidder-optimal core-selecting (BOCS) auctions, which are in turn a subset of the equilibria of the Vickrey auction. However, the equilibrium outcomes of the Vickrey and BOCS auctions are the same and include *any* assignment and payments that generate individually rational payoffs. The PAB auction rules out some, but not all, inefficient outcomes. In all auctions, we find equilibria in which the seller has no incentive to manipulate the auction after collecting bids and equilibria in which budget constraints are never binding. We also construct an extended mechanism that implements the tie-breaking rule.

Key words: Combinatorial auction, core-selecting auction, pay-as-bid auction, VCG mechanism, Nash equilibrium, seller incentives, tie breaking

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1. Introduction

Combinatorial or package auctions are increasingly used to sell heterogenous items because they give bidders the opportunity to express preferences involving substitutable and complementary goods. The complex strategy space that includes bids for all bundles of items – 2^k bids when there are k distinct items in the auction – makes equilibria difficult to analyze and outcomes hard to predict. Several studies find bids that form equilibria of combinatorial auctions, but aside from Bernheim and Whinston (1986), there has not been, to the best of our knowledge, a full characterization of all of the Nash equilibria of a combinatorial auction in a general setting with multiple, heterogenous goods.

This paper provides a systematic analysis and comparison of *all* pure-strategy, full-information, Nash equilibria of a class of sealed-bid combinatorial auctions that includes all core-selecting auctions and the Vickrey auction. The class contains any auction that assigns bundles to maximize the sum of bids and that chooses bidder payments between the reported social opportunity cost for the bundle won and the bid amount. Our characterization of best responses and equilibria traces the bidding incentives in and Nash equilibria of all auctions in this class back to two simple and intuitive conditions. The first says that each bidder wins the bundle of items that he would have won by bidding truthfully. The second says that each bidder pays his Vickrey payment, that is, the reported externality that he imposes on his opponents by winning these items. These two conditions emphasize strategic similarities and differences among the sealed-bid, combinatorial auctions in the class. The first condition is the same for all of the auctions in the class because each of these auctions assigns winning bundles in the same way. The second condition can take on a different meaning in different auctions because the bids that result in Vickrey payments depend on the auction's payment rule.

Characterizing the best responses and equilibria by these two conditions allows for straightforward comparisons among auctions. We find that the set of equilibria of the pay-as-bid auction is a (potentially strict) subset of that of any other auction in our class, including all core-selecting auctions and the Vickrey auction. All bidder-optimal core-selecting auctions have the same equilibria and this set of equilibria is a (potentially strict) subset of that of the Vickrey auction. So the Vickrey auction admits the most equilibria in the class, though all but one are in weakly dominated strategies. Equilibrium bids of any auction in the class will lead to the same outcome as if the Vickrey auction rules were applied to those equilibrium bids. This implies that, in all the equilibria of all core-selecting auctions, the bidders report values such that the Vickrey payoffs are in the core with respect to the bids.

In addition to analyzing the Nash equilibria of these auctions, we characterize the range of the resulting equilibrium outcomes – assignments, payments, and payoffs – for the pay-as-bid auction and for a second class of auctions that is a subset of the first class containing the bidder-optimal core-selecting auctions and the Vickrey auction. For this second class of auctions, we prove a type of folk theorem about the equilibrium outcomes. Any feasible assignment that does not leave valuable goods unassigned coupled with payments that correspond to individually rational payoffs forms an outcome of a full-information Nash equilibrium of every auction in the class. Thus, virtually any outcome is possible in equilibrium. Core-selecting auctions were introduced by Day and Milgrom (2008), in part, to deal with low revenue outcomes of the Vickrey auction.[1] However, our result shows that the bidder-optimal core-selecting auctions have the same equilibrium outcomes as the Vickrey auction, including those with low seller revenues.

The equilibrium outcomes of the pay-as-bid auction must satisfy a stronger necessary and sufficient condition than individual rationality, but we show that even the pay-as-bid auction can lead to inefficient assignments and payments below the Vickrey payments that result from truthful bidding. This wide range of outcomes suggests that conclusions based on particular Nash equilibria of the Vickrey auction or any core-selecting auction should be interpreted with caution.

Most of the literature that derives full-information Nash equilibria of combinatorial auctions considers equilibria where a player bids some fixed constant below his true value (or zero) for every bundle. They call these strategies truthful, semi-sincere, profit-target, or truncation

[1] See Ausubel and Milgrom (2006) for a detailed discussion of the weaknesses of the Vickrey auction.

strategies (Bernheim and Whinston, 1986; Ausubel and Milgrom, 2002; Milgrom, 2004; Day and Raghavan, 2007; Day and Milgrom, 2008). Bernheim and Whinston (1986) characterize all of the Nash equilibria of menu auctions and provide a refinement that selects equilibria in truncation strategies. A special case of their model is the allocation of multiple, heterogenous goods using one of the auctions we consider: the pay-as-bid auction. Ausubel and Milgrom (2002) and Day and Milgrom (2008, 2013) find that these same Nash equilibria in truncation strategies are equilibria of every core-selecting auction. Sano (2010) shows that these equilibria are also equilibria of the Vickrey auction (i.e., the VCG mechanism) and Sano (2013) generalizes this result by showing that a specific class of auctions, which Sano calls the Vickrey-reserve auctions, all share these Nash equilibria in truncation strategies while auctions that allow payments below Vickrey payments do not.

Equilibria that do not involve truncation strategies have also been identified in combinatorial auctions. For sealed-bid combinatorial Vickrey auctions, it is well known that truthful bidding is a weakly dominant strategy. Additionally, Holzman and Monderer (2004) and Holzman et al. (2004) analyze strategies that satisfy a non-standard, recursive definition of an ex-post equilibrium. For the special cases of single-unit or multiple-unit Vickrey auctions, where bidders have weakly decreasing marginal values, Blume and Heidhues (2004) and Blume et al. (2009) characterize the sets of ex-post equilibria. Beck and Ott (2013) find Nash equilibria of core-selecting auctions with weak overbidding on all bundles. Limited results about iterative combinatorial auctions also exist: Bichler et al. (2013) find ex-post equilibria of a modified combinatorial clock auction, Bikhchandani et al. (2011) prove that truthful bidding is an ex-post equilibrium of a particular ascending Vickrey auction, and Sano (2012) identifies a perfect Bayesian equilibrium of an ascending core-selecting auction in which a small bidder quits early to free ride on a complementary bidder. However, aside from Bernheim and Whinston (1986), there has not been a comprehensive characterization of all Nash equilibria of a core-selecting auction or the Vickrey auction with unrestricted bidder values. Even Bernheim and Whinston (1986) derive equilibrium *outcomes* only for particular equilibrium refinements and do not make general comparisons of outcomes across auctions.[2]

The paper proceeds as follows. Section 2 presents the model. Section 3 characterizes the best responses and the Nash equilibria of our class of combinatorial auctions and Section 4 analyzes the potential equilibrium outcomes in terms of assignments, payments, and payoffs. In Section 5, the results of the previous sections are used to derive insights into the seller's incentives and settings with budget constraints. Section 6 concludes. The tie-breaking rule used to obtain our main results and a possible mechanism for implementing it are discussed in Appendix A.

2. Model

We consider a setting with one seller, whom we denote 0, and a set of bidders $N = \{1, \ldots, n\}$, with $n \geq 2$. The seller owns a set of goods $K = \{1, \ldots, k\}$, with $k \geq 1$, which he does not value. Each bidder i has values $v_i : 2^K \to \mathbb{R}_+$ for bundles of goods $y \in 2^K$. We normalize $v_i(\emptyset) = 0$ for all $i \in N$ and we denote the vector of these values $v = (v_1, \ldots, v_n)$. Bidders have quasilinear utility, so if bidder i wins bundle y and pays price p_i, then he gets a payoff of $v_i(y) - p_i$. The seller's payoff is the sum of the payments made by the bidders.

Bidders place bids $b = (b_1, \ldots, b_n)$, where the vector of bidder i's bids for all bundles is $b_i = (b_i(y))_{y \in 2^K}$. Let b_S and b_{-S} be the vectors of bids of groups $S \subseteq N$ and $N \backslash S$, respectively. Let $B = B_1 \times \cdots \times B_n$, where B_i denotes the set of feasible bids for bidder i and $B_i = \mathbb{R}_+^{2^K}$ with the normalization $b_i(\emptyset) = 0$.

A sealed-bid, combinatorial auction is a direct mechanism that assigns each bidder a bundle of goods $x_i(b)$ and a payment $p_i(b, x(b))$ based on bids. Let $x_0(b) = K \setminus \{\cup_{i \in N} x_i(b)\}$ denote any unassigned items and $x(b) = (x_0(b), \ldots, x_n(b))$. The set of feasible assignments of goods $L \subseteq K$ is

$$X(L) = \left\{ x = (x_0, \ldots, x_n) \mid x_i \in 2^L \ \forall i \in \{0, \ldots, n\}, x_i \cap x_j = \emptyset \ \forall i \neq j, \cup_{i=0}^{n} x_i = L \right\}.$$

[2]For auctions with two bidders, Bernheim and Whinston (1986) find that their Nash equilibrium refinement selects outcomes that are equivalent to the outcomes of the Vickrey auction resulting from the Nash equilibrium in weakly dominant strategies.

The realized payment $p_i(b, x(b))$ depends on the assignment as well as the bids because when randomization is used to break ties, the payment will change with the realized assignment $x(b)$. Apart from randomization used to break ties, we consider only deterministic mechanisms.

2.1. Class $\mathcal{A}$ Auctions

To describe the class of auctions studied in this paper, we must first define the coalitional function w and the corresponding set of optimal assignments $\hat{X}$.

$$w(b) = \max_{x \in X(K)} \sum_{i \in N} b_i(x_i)$$

$$\hat{X}(b) = \arg\max_{x \in X(K)} \sum_{i \in N} b_i(x_i)$$

We call w the coalitional function because it is the maximum reported value that the coalition of the bidders and the seller can generate by trading the goods in K. To represent the reported value generated by the coalition of some subset of bidders $S \subseteq N$ and the seller or the reported value generated using only some subset of the seller's goods $L \subseteq K$, we use the following notation.

$$w(b_S^L) = \max_{x \in X(L)} \sum_{i \in S} b_i(x_i)$$

$$\hat{X}(b_S^L) = \arg\max_{x \in X(L)} \sum_{i \in S} b_i(x_i)$$

Likewise, $w(b_{-S}^{-L})$ represents the reported value generated by bidders $N \setminus S$ and goods $K \setminus L$. We omit the index whenever we refer to the full set N or K, e.g., $w(b_{-i}) = w(b_{-i}^K)$.

All auctions studied in this paper select an *optimal* (i.e., reported-value-maximizing) assignment in $\hat{X}(b)$. However, such optimal assignments may not maximize the sum of the bidders' true values. We call any value-maximizing assignment $x \in \hat{X}(v)$ *efficient* because, in our auction games, such an assignment corresponds to efficient payoffs.

There may be multiple optimal assignments, so in the main text we use a particular *tie-breaking rule* to select the assignment when $\hat{X}(b)$ is not a singleton: we break ties in favor of best responses. In other words, the tie-breaking rule chooses the $x \in \hat{X}(b)$ such that conditions

(I) and (II), as defined in Section 3, are satisfied for as many $i \in N$ as possible. If there is more than one such assignment, the auction selects each with equal probability. Using this particular tie-breaking rule eases notation, but most of our results hold for any possible method of breaking ties, as detailed further in Appendix 6. We break ties in favor of best responses in the main text to maximize the number of Nash equilibria. If ties are broken differently, some of the Nash equilibria we characterize may be destroyed, but no new equilibria will be added.[3] Appendix 6 proves this assertion, discusses the implications of using other tie-breaking rules, and shows how to extend the game to endogenously implement tie breaking in favor of best responses.

Payments assessed by the auctions we study are bounded above by the bid amount and bounded below by the minimum bid necessary for the bidder to win his assigned bundle. This minimum bid equals the reported opportunity cost that a bidder's presence imposes on the other bidders and is the payment in one famous auction in our class, the Vickrey auction.

Definition 1. *The* Vickrey auction *selects $x(b) \in \hat{X}(b)$, breaks ties in favor of best responses, and chooses $p_i^V(b, x(b)) = w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b))$ for all $i \in N$.*

Now that we have defined optimal assignments and Vickrey payments, we are ready to describe the class of sealed-bid, combinatorial auctions considered in our main results on best responses and equilibria.

Definition 2. *The class of auctions $\mathcal{A}$ consists of all sealed-bid auctions $(x(b), p(b, x(b)))$ that, for every $b \in B$, select $x(b) \in \hat{X}(b)$, break ties in favor of best responses, and choose payments $p_i(b, x(b)) \in \left[p_i^V(b, x(b)), b_i(x_i(b))\right]$ for all $i \in N$.*

Class $\mathcal{A}$ contains infinitely many auctions that choose an optimal assignment and select payments no higher than bids and no lower than the minimum bid necessary to win the assigned

[3]Since our tie-breaking rule maximizes the set of Nash equilibria, it is the least restrictive rule and the most appropriate for our goal of characterizing all Nash equilibria. While the set of Nash equilibria depends on the tie-breaking rule, most of our results hold for all possible rules. Appendix 6 lists the theorems that do not depend on the tie-breaking rule and explains changes needed for the other theorems to work with different tie-breaking rules.

bundle.[4] Notable auctions in this class include the Vickrey auction and so-called *core-selecting* auctions.

2.2. Core-Selecting Auctions

Class $\mathcal{A}$ consists of more than just core-selecting auctions, but we define this particular subclass because (i) it includes auctions addressed by the previous literature and (ii) our characterization of equilibria allows us to make some conclusions related to specific core-selecting auctions. Core-selecting auctions use the core, a concept from cooperative game theory, as part of their design. The core $\mathcal{C}(v)$ of a cooperative game consists of all feasible *payoffs* that are not blocked by any coalition (i.e., each group receives at least as much as it could achieve on its own so that it could not deviate and make all of its members better off). In our setting, any group that does not include the seller cannot generate any value because the seller owns all of the goods. Any coalition that contains the seller must receive at least what it could get from trading among its members. Therefore, the core consists of all payoff vectors $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ that satisfy the following constraints (where the first is feasibility and the rest ensure no coalition can block the payoffs):

$$\begin{aligned} \pi_0 + \textstyle\sum_{i \in N} \pi_i &\leq w(v) \\ \pi_0 &\geq 0 \\ \pi_i &\geq 0 \qquad \forall i \in N \\ \pi_0 + \textstyle\sum_{i \in S} \pi_i &\geq w(\{v_i\}_{i \in S}) \quad \forall S \subseteq N \end{aligned}$$

In our setting with a single seller, the core is always nonempty. For example, the payoffs $\pi_0 = w(v)$ and $\pi_i = 0$ for all $i \in N$ satisfy these constraints for all v.

A core-selecting auction is a direct mechanism (inducing a non-cooperative game among bidders) that maps bids on bundles of goods to assignments and payments such that *reported*

[4] Ignoring the tie-breaking rule, class $\mathcal{A}$ consists of the *Vickrey-reserve auctions* defined by Sano (2013) as efficient and individually rational auctions with payments above Vickrey payments. Sano (2013) calls these auctions efficient because they choose an optimal assignment based on bids and he calls them individually rational because payments are below bid amounts.

payoffs are in the core with respect to the bids, $\mathcal{C}(b)$. Denote bidder i's reported payoff as $\pi_i^r(b) = b_i(x_i(b)) - p_i(b, x(b))$ and the seller's payoff as $\pi_0^r(b) = \sum_{i \in N} p_i(b, x(b))$. Note that the seller actually receives these payments from the bidders, so $\pi_0^r(b)$ is the same as the seller's true payoff, $\pi_0(b)$, and equals the revenues he receives. However, a bidder's true payoff resulting from bids b, $\pi_i(b) = v_i(x_i(b)) - p_i(b, x(b))$, need not equal his reported payoff $\pi_i^r(b)$.

Definition 3. *A* core-selecting auction *is a direct mechanism that breaks ties in favor of best responses and chooses $x(b)$ and $p(b, x(b))$ such that:*

$$\pi^r(b) = (\pi_0^r(b), \pi_1^r(b), \ldots, \pi_n^r(b)) \in \mathcal{C}(b) \quad \forall b \in B$$

All core-selecting auctions choose an optimal assignment $x(b) \in \hat{X}(b)$, which is necessitated by the first and last core inequalities, as given above. However, there may be infinitely many possible payments that satisfy the core constraints for any given optimal assignment and, therefore, there are infinitely many core-selecting auctions. Translating the constraints on payoffs into constraints on the assignment and payments yields:

$$\begin{aligned}
& \pi_0^r(b) + \sum_{i \in N} \pi_i^r(b) = w(b) && \Leftrightarrow && x(b) \in \hat{X}(b) && (1)\\
& \pi_0^r(b) \geq 0 && \Leftrightarrow && \sum_{i \in N} p_i(b, x(b)) \geq 0 \\
\forall i \in N: \quad & \pi_i^r(b) \geq 0 && \Leftrightarrow && b_i(x_i(b)) \geq p_i(b, x(b)) \\
\forall S \subset N: \quad & \pi_0^r(b) + \sum_{i \in S} \pi_i^r(b) \geq w(b_S) && \Leftrightarrow && \sum_{j \in N \setminus S} p_j(b, x(b)) \geq w(b_S) - \sum_{i \in S} b_i(x_i(b))
\end{aligned}$$

The second constraint is implied by the last constraint: $p_i(b, x(b)) \geq w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b))$ ≥ 0 prevents negative payments – otherwise the seller and the bidders in $N \setminus \{i\}$ could do better – and implies $\sum_{i \in N} p_i(b, x(b)) \geq 0$. Also, the third constraint implies that losing bidders pay zero.

Some particular core-selecting auctions are the bidder-optimal core-selecting auctions and the pay-as-bid auction. Bidder-optimal core-selecting auctions select payoffs on the bidder-

Pareto-optimal frontier of the core.

Definition 4. *A* bidder-optimal core-selecting (BOCS) auction *is a core-selecting auction that chooses $x(b)$ and $p(b, x(b))$ such that, for all $b \in B$, there does not exist any $\hat{\pi}^r \in \mathcal{C}(b)$ such that $\hat{\pi}_i^r \geq \pi_i^r(b)$ for all $i \in N$ and the inequality is strict for at least one $i \in N$.*

There are generally many bidder-optimal payments, so a full specification of a particular BOCS auction requires a rule for choosing between them. For example, a payment rule might choose the bidder-optimal payment vector that is closest in Euclidean distance to the Vickrey payments, as suggested by Day and Cramton (2012). On the opposite end of the spectrum is the core-selecting auction that maximizes the seller's revenues among reported core payoffs by having each bidder pay his full bid for his winning bundle.[5] This *pay-as-bid* auction is the combinatorial version of a first-price auction.

Definition 5. *The* pay-as-bid (PAB) auction *selects $x(b) \in \hat{X}(b)$, breaks ties in favor of best responses, and chooses $p_i(b, x(b)) = b_i(x_i(b))$ for all $i \in N$.*

The conditions of class $\mathcal{A}$ are satisfied by every core-selecting auction, including the PAB auction and all BOCS auctions. The third core constraint says that bids weakly exceed payments and the last core constraint requires that each bidder's payment be at least as large as his Vickrey payment, $p_i^V(b, x(b))$. This also implies that each bidder's reported payoff in any core-selecting auction is no larger than his reported Vickrey payoff: $\pi_i^{r,V}(b) := b_i(x_i(b)) - p_i^V(b, x(b)) = w(b) - w(b_{-i})$. The Vickrey payoff, in contrast to the Vickrey payment, does not depend on which optimal assignment is chosen in the case of ties.

Given bids b, there always exists a core allocation in which any single bidder i receives his reported Vickrey payoff because the payoff profile $\pi_0^r = w(b_{-i})$, $\pi_i^r = w(b) - w(b_{-i})$, and $\pi_j^r = 0$ for all $j \neq i$, always lies in the core $\mathcal{C}(b)$. However, it may not be possible for all bidders to simultaneously receive their reported Vickrey payoffs in a core allocation. When the reported Vickrey payoff vector $\pi^{r,V}(b) = \left(\pi_0^{r,V}(b), \pi_1^{r,V}(b), \ldots, \pi_n^{r,V}(b)\right)$ with $\pi_0^{r,V}(b) =$

[5] The bidder-pessimal core allocation is always unique: $(\pi_0^r(b), \pi_1^r(b), \ldots, \pi_n^r(b)) = (w(b), 0, \ldots, 0)$.

$w(b) - \sum_{i \in N} \pi_i^{r,V}(b)$ lies in the core, it is the unique bidder-optimal allocation and the unique outcome of any BOCS auction. When it is not in the core, there are multiple bidder-optimal allocations, all of which give the seller a larger payoff than $\pi_0^{r,V}(b)$.[6]

Any auction that, for some profile of bids, awards the seller revenue between his Vickrey payoff and his lowest payoff from a BOCS auction is not core-selecting. However, such an auction does belong to class $\mathcal{A}$ and will have similar equilibria to those of core-selecting auctions, as shown in the next section.

3. Nash Equilibria

In this section, we characterize all of the pure-strategy, full-information, Nash equilibria of the combinatorial auctions in class $\mathcal{A}$. We prove that such equilibria exist for all v and compare equilibria across auctions in $\mathcal{A}$. Their equilibria are similar because, holding constant the bids of his opponents, a bidder has the same maximum achievable payoff in all of these auctions due to the shared lower bound on payments. This maximum payoff results from winning something he would have won by bidding truthfully and paying his Vickrey payment for it. Moreover, in all of these auctions, there is a common best-response bid that guarantees this maximum payoff.

The following conditions, which ensure a bidder this maximum payoff, are essential to our main results on Nash equilibria:

(I) $x(b) \in \hat{X}(v_i, b_{-i})$ for all $x(b)$ chosen with positive probability

(II) $p_i(b, x(b)) = p_i^V(b, x(b)) = w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b))$ for all $x(b)$ chosen with positive probability

(II$'$) $w(b) = w(b_{-i})$

Condition (I) guarantees that bidder i wins what he would win if he reported his values truthfully, holding the other bids fixed: $x(b) \in \arg\max_{x \in X(K)} \left[v_i(x_i) + \sum_{j \in N \setminus \{i\}} b_j(x_j) \right]$. Condition (II) ensures that bidder i pays as little as possible for the bundle he wins because payments

[6]See Ausubel and Milgrom (2002), Theorem 6, and Ausubel and Milgrom (2006), Theorem 5.

are bounded below by Vickrey payments in all of the auctions that we consider and, as we will show, the Vickrey payment is achievable in all of these auctions. When bids are such that the tie-breaking rule randomizes among multiple assignments, these first two conditions must hold for all assignments chosen with positive probability to maximize bidder i's *ex ante* expected payoff. Finally, condition (II′) is a special case of (II) when $p_i(b, x(b)) = b_i(x_i(b))$. This condition will apply to the PAB auction because this auction selects payments equal to winning bids.

The conditions above are necessary and sufficient for a bid b_i to be a best response in some auction in $\mathcal{A}$.

Theorem 1. *Consider any profile of bids b_{-i}.*

(a) *For every auction in $\mathcal{A}$, b_i is a best response to b_{-i} if and only if (I) and (II) hold.*

(b) *In the PAB auction, b_i is a best response to b_{-i} if and only if (I) and (II′) hold.*

(c) *If b_i is a best response to b_{-i} in the PAB auction, then b_i is a best response to b_{-i} in every auction in $\mathcal{A}$.*

Proof: (a) Consider bidder i and fix all other bids b_{-i}. In every auction $A \in \mathcal{A}$, bidder i's payoff is bounded above by the true value he adds to the auction, given b_{-i}: $w(v_i, b_{-i}) - w(b_{-i})$. For every $b_i \in B_i$,

$$
\begin{aligned}
v_i(x_i(b)) - p_i(b, x(b)) &\leq v_i(x_i(b)) - p_i^V(b, x(b)) && (2)\\
&= v_i(x_i(b)) + \sum_{j \neq i} b_j(x_j(b)) - w(b_{-i}) && (3)\\
&\leq \max_{z \in B_i} \left[v_i(x_i(z, b_{-i})) + \sum_{j \neq i} b_j(x_j(z, b_{-i})) \right] - w(b_{-i}) && (4)\\
&= v_i(x_i(v_i, b_{-i})) + \sum_{j \neq i} b_j(x_j(v_i, b_{-i})) - w(b_{-i}) && (5)\\
&= w(v_i, b_{-i}) - w(b_{-i}) && (6)
\end{aligned}
$$

The inequality in (2) holds by the definition of class $\mathcal{A}$. Equalities (3) and (6) hold by the definitions of Vickrey payments and the coalitional function. The inequality in (4) is by maximization and (5) holds because all auctions in $\mathcal{A}$ choose an optimal (or bid maximizing) assignment.

The inequality in (2) holds with equality for all $x(b)$ chosen with positive probability if and only if (II) holds. The inequality in (4) holds with equality for all $x(b)$ chosen with positive probability if and only if (I) holds, by definition of $\hat{X}(v_i, b_{-i})$. Therefore, bidder i's payoff equals the upper bound with probability one if and only if (I) and (II) hold.

A bidder can get arbitrarily close to the upper bound by bidding zero for every package except what he would win if he bid truthfully and placing a bid on that package of $b_i(x_i(v_i, b_{-i})) = w(b_{-i}) - \sum_{j\neq i} b_j(x_j(v_i, b_{-i})) + \varepsilon$. This bid guarantees that $x(v_i, b_{-i}) \in \hat{X}(b)$ is one of the optimal assignments if $\varepsilon = 0$ and is the unique optimal assignment (i.e, $\{x(v_i, b_{-i})\} = \hat{X}(b)$) if $\varepsilon > 0$. When bidder i wins $x_i(v_i, b_{-i})$, his payment must be within ε of the Vickrey payment because his bid, which is the upper bound on his payment for any auction in class $\mathcal{A}$, equals $p_i^V(b, x(v_i, b_{-i})) + \varepsilon$. Therefore, if the tie is broken in bidder i's favor and the auction chooses $x(v_i, b_{-i})$ with probability one, then this bid with $\varepsilon = 0$ generates exactly the upper bound on his payoff. When the tie is not broken in his favor, this bid guarantees him a payoff within ε of the upper bound for every $\varepsilon > 0$.

Since bidder i can get a payoff arbitrarily close to this upper bound for all b_{-i} and all $A \in \mathcal{A}$, b_i is a best response if and only if (I) and (II) are satisfied.

(b) For the PAB auction, (II) holds if and only if (II′) holds because the PAB auction has payment rule $p_i(b, x(b)) = b_i(x_i(b))$ and (II′) is independent of the optimal assignment $x(b)$. The result then follows directly from (a).

(c) Condition (I) means the same thing for all auctions in $\mathcal{A}$. If (II′) holds for any given b, then $p_i^V(b, x(b)) = w(b_{-i}) - \sum_{j\neq i} b_j(x_j(b)) = b_i(x_i(b))$ for all optimal assignments $x(b)$. Therefore, $p_i(b, x(b)) = p_i^V(b, x(b)) = b_i(x_i(b))$ and satisfies (II) for every auction in $\mathcal{A}$. ■

Best responses in all auctions in $\mathcal{A}$ must satisfy the same conditions. However, because condition (II) depends on the auction's particular payment rule, Theorem 1(a) does not imply that all auctions have the same best responses. Theorem 1(c) shows that they share some common best responses, those which satisfy conditions (I) and (II′). These two conditions are independent of the auction's payment rule.

The previous literature focused on *truncation strategies* as best replies: $b_i(y) = \max\{v_i(y) - [w(v_i, b_{-i}) - w(b_{-i})], 0\}$ for all $y \in 2^K$. Bernheim and Whinston (1986) first proved that a bidder always has a truncation strategy among his best responses in the PAB auction and hence that the PAB auction has equilibria in which all bidders play truncation strategies. Day and Milgrom (2008) extended the result to all core-selecting auctions, and Sano (2013) later extended it to all auctions in $\mathcal{A}$.[7] These truncation strategies satisfy (I) and (II) for all $A \in \mathcal{A}$, so Theorem 1 confirms that they are best responses. However, infinitely many other bidding strategies also satisfy (I) and (II). The proof of Theorem 1 contains one example.[8] Moreover, not only the truncation strategies, but all best responses in the PAB auction, are best responses in all auctions in $\mathcal{A}$.

Best responses need not involve bidding below a bidder's true values. Beck and Ott (2013) demonstrate that a bidder always has an *overbidding strategy*, one with $b_i(y) \geq v_i(y)$ for all $y \in 2^K$, among his best responses in every BOCS auction. The strategy used in their proof fulfills conditions (I) and (II$'$). Thus, in all auctions in $\mathcal{A}$, bidders have best responses in both overbidding and truncation strategies.[9]

This leads to our main theorem about Nash equilibria.

Theorem 2. *Consider any auction $A \in \mathcal{A}$. Bids b^* form a Nash equilibrium of auction A if and only if (I) and (II) hold for all $i \in N$. For every v, a pure-strategy Nash equilibrium b^* exists.*

Proof: From Theorem 1(a), it follows directly that no bidder has a profitable deviation and, therefore, that b^* is a Nash equilibrium if and only if (I) $x(b^*) \in \hat{X}(v_i, b^*_{-i})$ and (II) $p_i(b^*, x(b^*)) = p_i^V(b^*, x(b^*))$ for all $x(b^*)$ chosen with positive probability and all $i \in N$.

[7] These authors use different tie-braking rules than we use here. Bernheim and Whinston (1986) break ties in favor of efficiency while Day and Milgrom (2008) and Sano (2013) break ties in favor of *favored bidders*.

[8] For the bundle that bidder i wins, his bid in the proof of Theorem 1 is equivalent to that of a truncation strategy, but his bids for all other bundles differ because they are zero.

[9] In core-selecting auctions, both types of strategies may be dominated or undominated. Beck and Ott (2013) consider settings in which overbidding strategies are undominated and truncation strategies are dominated in BOCS auctions. In PAB auctions, overbidding strategies are dominated and truncation strategies may be undominated. All non-truthful strategies are weakly dominated in the Vickrey auction.

To prove that a Nash equilibrium exists for any given v, we will show that the following bids b^* satisfy conditions (I) and (II) for all $i \in N$. Define bids b^* as follows:

$$b_i^*(y) = v_i(y) \qquad \forall i \in N,\ \forall y \subset K$$
$$b_i^*(K) = w(v) \qquad \forall i \in N$$

Breaking ties in favor of best responses means favoring the assignment that satisfies (I) for all $i \in N$, so a bidder can be assigned K only if $v_i(K) = w(v)$. Since bids for all bundles except K are truthful, $v_i(x_i(b^*)) + \sum_{j \neq i} b_j^*(x_j(b^*)) = \sum_{j \in N} b_j^*(x_j(b^*)) = w(b^*) = w(v) = w(v_i, b_{-i}^*)$ for all $i \in N$ and for any $x(b^*)$ chosen with positive probability. Thus, any $x(b^*)$ chosen with positive probability generates value $w(v_i, b_{-i}^*)$ and condition (I) holds for all $i \in N$.

Furthermore, by construction, $w(b^*) = w(v) = w(b_{-i}^*)$ for all $i \in N$. This implies that $p_i^V(b^*, x(b^*)) = b_i(x_i(b^*))$ for every $x(b^*)$ chosen with positive probability, which collapses the payment interval to a single point for all bidders and, thereby, ensures condition (II) holds for all $i \in N$. ■

Having bids satisfy conditions (I) and (II) for all $i \in N$ is both necessary and sufficient for equilibrium. If bids do not satisfy (I) or (II) for some bidder i, then that bidder can achieve a strictly higher payoff from bidding $b_i(y) = w(b_{-i}) - w(b_{-i}^{-y})$ for some y he would win if he bid truthfully and bidding zero for all other bundles. On the other hand, if bids satisfy condition (II) for all $i \in N$, then every bidder pays his Vickrey payment. Given that he pays his Vickrey payment, a bidder strictly prefers his payoff from any assignment $x \in \hat{X}(v_i, b_{-i})$ over any assignment $x \notin \hat{X}(v_i, b_{-i})$. Note that he receives the same payoff from every $x \in \hat{X}(v_i, b_{-i})$, given that he pays his Vickrey payoff. Therefore, conditions (I) and (II) holding for all $i \in N$ prevents any bidder from having a profitable deviation and ensures that bids form an equilibrium.

Given most valuation profiles, there exist infinitely many equilibria of the auctions in $\mathcal{A}$. Similar to the best responses in Theorem 1, these equilibria contain both overbidding and truncation strategies (e.g., Beck and Ott (2013); Day and Milgrom (2008)) and can differ across auctions. Condition (II) depends on the auction's payment rule, so different bids will satisfy (II) for different auctions. The next theorem compares the equilibria of some common combinatorial auctions.

Theorem 3.

(a) *b^* is a Nash equilibrium of the PAB auction if and only if (I) and (II′) hold for all $i \in N$.*

(b) *b^* is a Nash equilibrium of the Vickrey auction if and only if (I) holds for all $i \in N$.*

(c) *All BOCS auctions have the same Nash equilibria.*

Proof: **(a)** The PAB auction has payment rule $p_i(b, x(b)) = b_i(x_i(b))$, so for the PAB auction (II) holds if and only if (II′) holds. The result then follows directly from Theorem 2.[10]

(b) The Vickrey auction's payment rule dictates that (II) holds for all $b \in B$. Thus, condition (II) is unnecessary and the result follows directly from Theorem 2.

(c) According to condition (II), for some b^* to be an equilibrium of a BOCS auction, it must result in *all* bidders receiving their reported Vickrey payoffs. This can happen only if $\pi^{r,V}(b^*) \in \mathcal{C}(b^*)$, in which case it is the unique bidder-optimal allocation and every BOCS auction chooses the same payments. Since all BOCS auctions choose the assignment in the same way for a given b^*, condition (I) holds for some b^* in some BOCS auction if and only if it holds for that b^* in all BOCS auctions. Therefore, if b^* is a Nash equilibrium of some BOCS auction, it must be a Nash equilibrium of all BOCS auctions. ■

In the PAB auction, the bid $b_i(x_i(b))$ for the item that bidder i wins decides his payment. Thus, he must bid just high enough to win the bundle in equilibrium. In BOCS auctions, equilibrium bids can be higher because the auction rules will not always force bidders to pay their full bids. Both a bidder's winning bid and his losing bids can influence his payment in a BOCS auction. In the Vickrey auction, bidder i's bid determines only which bundle he wins and has no impact on his payment. For this reason, the Vickrey auction admits the most equilibria and the PAB auction the least.

[10] Bernheim and Whinston (1986) provided the first characterization of the full set of Nash equilibria of a pay-as-bid menu auction.

Theorem 4. *Denote by* $NE^A(v)$ *the set of Nash equilibria of auction* A *for a given profile of values* v*. Denote the PAB auction and the Vickrey auction by* $A = PAB$ *and* $A = V$*, respectively.*

(a) $NE^{PAB}(v) \subseteq NE^A(v) \subseteq NE^V(v)$ *for all* v *and for all* $A \in \mathcal{A}$*.*

(b) *If* $n > 2$*, there exists* v *such that* $NE^{PAB}(v) \subset NE^A(v) \subset NE^V(v)$ *for every BOCS auction* A*.*

(c) *If* $n = 2$*,* $NE^A(v) = NE^V(v)$ *for all* v *and all BOCS auctions* A*, and there exists* v *such that* $NE^{PAB}(v) \subset NE^A(v)$ *for every BOCS auction* A*.*

Proof: **(a)** Given bids, condition (I) means the same thing in every auction in $\mathcal{A}$. Additionally, given bids, condition (II′) implies condition (II) for every auction in $\mathcal{A}$ because $w(b) = w(b_{-i})$ implies $p_i^V(b, x(b)) = w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b)) = w(b) - \sum_{j \neq i} b_j(x_j(b)) = b_i(x_i(b))$ for all $x(b) \in \hat{X}(b)$. Therefore, the statement follows from Theorems 2, 3(a), and 3(b).

(b) To prove that both subsets may be strict for certain value profiles, consider the following example where the set of goods is $K = \{C, D\}$ and the non-empty bundles of goods are denoted C, D, and CD, respectively.

	C	D	CD
v_1	1	0	3
v_2	0	2	2
v_3	0	0	2

Truthful bidding is always an equilibrium of the Vickrey auction, but it is not an equilibrium of the PAB auction in this example because condition (II′) is not satisfied: $w(v) = 3 > 2 = w(v_{-1})$. It is also an equilibrium of every BOCS auction because (I) is trivially satisfied by $b = v$ and because $\pi^V(v) \in \mathcal{C}(v)$, so every BOCS auction charges bidders their Vickrey payments and (II) holds. Thus, $\mathrm{NE}^{PAB}(v) \subset \mathrm{NE}^A(v)$ for every BOCS auction A.

For the values shown above, the following bids form an equilibrium of the Vickrey auction because condition (I) holds, but they do not form an equilibrium of any BOCS auction. With the bids below, bidders 1 and 2 each win one item. Every BOCS auction requires $p_1 + p_2 = 2$, which implies condition (II) cannot hold for both 1 and 2. Either $p_1 > w(v_{-1}) - b_2(x_2(b)) = 0$ or $p_2 > w(v_{-2}) - b_1(x_1(b)) = 0$. Thus, $\text{NE}^A(v) \subset \text{NE}^V(v)$ for every BOCS auction A.

	C	D	CD
b_1	2	0	2
b_2	0	2	2
b_3	0	0	2

(c) When $n = 2$, $\text{NE}^A(v) = \text{NE}^V(v)$ for all v because the Vickrey and BOCS auctions choose the assignment and payments in the same way. BOCS auctions choose payments for bidders $i \in N = \{1, 2\}$ such that $p_i \geq p_i^V(b, x(b))$ and no Pareto-better (lower) payments exist. This amounts to choosing $p_i = p_i^V(b, x(b))$ for all $i \in N$ and all $b \in B$.

To prove there exist value profiles such that $\text{NE}^{PAB}(v) \subset \text{NE}^A(v)$ for every BOCS auction A, consider the following example where the set of goods is $K = \{C, D\}$.

	C	D	CD
v_1	1	0	1
v_2	0	1	1

Truthful bidding is always an equilibrium of the Vickrey auction, and thus of all BOCS auctions, but it is not an equilibrium of the PAB auction in this example because condition (II′) is not satisfied: $w(v) = 2 > 1 = w(v_{-1})$. ■

This theorem implies that analyzing the equilibria of the PAB auction will shed light on all other auctions in $\mathcal{A}$ because they share its equilibria. Moreover, the set of equilibria of the PAB auction is the largest set that all auctions in $\mathcal{A}$ have in common. This result connects the findings from the literature on equilibria in the truncation strategies discussed above. In addition, it reveals that it is not only the equilibria in truncation strategies that are shared by all auctions in $\mathcal{A}$, but every equilibrium of the PAB auction.

Corollary 1. *All auctions in $\mathcal{A}$ have Nash equilibria in common: b^* is an equilibrium of every auction in $\mathcal{A}$ if and only if (I) and (II′) hold for all $i \in N$.*

Theorem 4 shows the relationship between the equilibria of specific auctions for given value profiles. In particular, though these auctions have equilibria with similar properties, the set of equilibria of the PAB auction can be strictly smaller than that of BOCS auctions, and likewise for BOCS auctions and the Vickrey auction. However, to understand whether the exclusion of some equilibria matters, we must consider the outcomes of these equilibria and not just the strategies themselves. The equilibrium outcomes are the subject of the next section.

4. Equilibrium Assignments, Payments, and Payoffs

This section studies the assignments, payments, and payoffs that result from equilibria of auctions in class $\mathcal{A}$. We start by comparing outcomes from the same equilibrium bids across the different auction rules. Due to condition (II), outcomes for equilibria shared among auctions in class $\mathcal{A}$ will be identical. We then examine the effect of the PAB auction having a (sometimes strictly) smaller set of equilibria than that of BOCS or Vickrey auctions.

In any auction in $\mathcal{A}$, equilibrium payments equal the Vickrey payments by condition (II). All of these auctions also choose the assignment in the same way. Therefore, whenever they share an equilibrium b^*, they produce the same results in terms of the assignment, payments, and payoffs. This shared equilibrium outcome equals that of the Vickrey auction rules applied to b^*.[11]

Theorem 5. *The assignment $x(b^*)$ and the payments $p(b^*, x(b^*))$ resulting from any Nash equilibrium b^* of any auction in $\mathcal{A}$ equal the assignment and payments calculated by applying the Vickrey auction rules to b^*.*

Proof: All auctions in $\mathcal{A}$, including the Vickrey auction, choose the assignment in the same way. By Theorem 2, any equilibrium b^* satisfies condition (II) for all $i \in N$, so $p_i(b^*, x(b^*)) = p_i^V(b^*, x(b^*))$ for all $x(b^*)$ chosen with positive probability and all $i \in N$. ■

[11] A similar result applies to best responses. For a given b_{-i}, if bidder i has the same best response in multiple auctions in $\mathcal{A}$, then he receives the same payoff of $\pi_i^V(v_i, b_{-i})$ from his best response bid in each of those auctions. This follows from Theorem 1 and the arguments in the proof of Theorem 5.

As a practical application, this result allows one to easily exclude bids from being an equilibrium of an auction in $\mathcal{A}$ without any information about v. If the bids result in different payments in some auction in class $\mathcal{A}$ than they would in the Vickrey auction, then those bids do not form an equilibrium of that auction.[12]

The theorem also implies that in any full-information Nash equilibrium of any core-selecting auction, the reported Vickrey payoffs will belong to the reported core (i.e., $\pi^{r,V}(b^*) \in \mathcal{C}(b^*)$), forming the unique bidder-optimal point in that core. Furthermore, in every equilibrium, the bidders will earn their true Vickrey payoffs, $v_i(x_i(b^*)) - p_i^V(b^*, x(b^*))$, which may be neither in the reported core $\mathcal{C}(b^*)$ nor in the core with respect to the true values $\mathcal{C}(v)$. This counteracts the reason for the development of core-selecting auctions in the first place: to correct some weaknesses of the Vickrey auction that occur precisely when the Vickrey payoffs are not in $\mathcal{C}(v)$ (see Ausubel and Milgrom, 2006).

We now turn to characterizing the precise outcomes that occur in equilibria of auctions in $\mathcal{A}$. For the PAB auction and a class of auctions that contains the Vickrey and BOCS auctions, we provide necessary and sufficient conditions for assignment-payment pairs to be supportable by some equilibrium. For ease of notation, we assume for the remainder of this section that bidders' values satisfy free disposal: $v_i(y) \leq v_i(y')$ for all $i \in N$ and all $y \subseteq y' \subseteq K$. All of the results will hold without free disposal, but one must modify the third condition in Theorems 6 and 7 to reflect the possibility that smaller sets of items yield larger values.

To determine the equilibrium assignments and payments, we will exploit outcome equivalence. The following lemma shows that any equilibrium outcome of the PAB auction can be achieved by an equilibrium with a simple bidding structure.

Lemma 1. *If b^* is an equilibrium of the PAB auction, then $\tilde{b}$ as defined below is also an*

[12] The reverse does not hold. For example, the bids $b_i(y) = 0$ for all y and all i lead to $p(b, x(b)) = p^V(b, x(b)) = (0, \ldots, 0)$ in every auction in $\mathcal{A}$, but these bids are not equilibria for generic v because unsuccessful bidders with positive values have profitable deviations.

equilibrium of the PAB auction and $\tilde{b}$ leads to the same assignment and payments as b^*.

$$\tilde{b}_i(x_i(b^*)) = b_i^*(x_i(b^*)) \quad \forall i \in N$$
$$\tilde{b}_i(K) = w(b^*) \quad \forall i \in N$$
$$\tilde{b}_i(y) = 0 \quad \forall i \in N, \forall y \notin \{x_i(b^*), K\}$$

Proof: We will show that $\tilde{b}$ satisfies conditions (I) and (II′) for all $i \in N$, so that by Theorem 3(a) it is a Nash equilibrium of the PAB auction. By construction, there are n or $n+1$ assignments that maximize the reported surplus: $x(b^*)$ and the assignments that give all of the goods to any single bidder. All of these assignments generate the same value, $w(b^*)$. So $w(b^*) = w(\tilde{b})$ and $w(b^*) = w(\tilde{b}_{-i})$ for all $i \in N$, which satisfies condition (II′).

To prove condition (I), first note that $w(v_i, b^*_{-i}) = \max_{x \in X(K)} \left[\sum_{j \neq i} b_j^*(x_j) + v_i(x_i)\right] \geq \max_{x \in X(K),\, x_j \neq K\, \forall j \neq i} \left[\sum_{j \neq i} b_j^*(x_j) + v_i(x_i)\right] \geq \max_{x \in X(K),\, x_j \neq K\, \forall j \neq i} \left[\sum_{j \neq i} \tilde{b}_j(x_j) + v_i(x_i)\right] \geq \sum_{j \neq i} \tilde{b}_j(x_j(b^*)) + v_i(x_i(b^*)) = \sum_{j \neq i} b_j^*(x_j(b^*)) + v_i(x_i(b^*)) = w(v_i, b^*_{-i})$. The first inequality holds because the maximum over a smaller set is weakly smaller. The second inequality holds because $\tilde{b}_j(y) \leq b_j^*(y)$ for all $y \neq K$. The third inequality holds because $x(b^*)$ is one feasible assignment. The next to last equality holds by construction of $\tilde{b}$ and the final equality holds because b^* is an equilibrium and thus fulfills (I).

Thus, $w(v_i, \tilde{b}_{-i}) = \max\left\{\max_{x \in X(K),\, x_j \neq K\, \forall j \neq i} \left[\sum_{j \neq i} \tilde{b}_j(x_j) + v_i(x_i)\right], \max_{j \neq i} \tilde{b}_j(K)\right\} = \max\left\{w(v_i, b^*_{-i}), w(b^*)\right\} = w(v_i, b^*_{-i})$. The second equality holds by the above inequalities and the third holds because $w(v_i, b^*_{-i}) = \sum_{j \neq i} b_j^*(x_j(b^*)) + v_i(x_i(b^*))$ by b^* fulfilling (I) and because we must have $v_i(x_i(b^*)) \geq b_i^*(x_i(b^*))$ in any equilibrium of the PAB auction. Therefore, because b^* is an equilibrium, $x(b^*)$ generates value $w(v_i, b^*_{-i}) = w(v_i, \tilde{b}_{-i})$, so $x(b^*) \in \hat{X}(v_i, \tilde{b}_{-i})$. Breaking ties in favor of best responses means favoring the assignment that satisfies $x(\tilde{b}) \in \hat{X}(v_i, \tilde{b}_{-i})$ for as many $i \in N$ as possible. One assignment that satisfies the condition for all $i \in N$ is $x(\tilde{b}) = x(b^*)$, so $\tilde{b}$ is an equilibrium and results in the same assignment and payments as b^* with positive probability. ■

This lemma allows us to reduce attention to equilibria with the simple structure above, in which bidders place nonzero bids only for the bundle they win and the bundle of all items in the auction. Bidders have fewer potentially profitable deviations against these simple bids than against a full set of nonzero bids, which helps us characterize the following necessary and sufficient condition for equilibrium outcomes of the PAB auction.

Theorem 6. *There exists a Nash equilibrium b^* of the PAB auction that results in outcome $x = x(b^*)$ and $p = p(b^*, x(b^*))$ if and only if the following conditions hold:*

(1) $x \in X(K)$

(2) $p_i \in [0, v_i(x_i)] \quad \forall i \in N$

(3) $\sum_{j\in S} p_j \geq \max_{i\in N\setminus S} [v_i (x_i \cup x_0 \cup (\cup_{j\in S} x_j)) - v_i(x_i)] \quad \forall S \subset N$

Proof: By Lemma 1, (x, p) will be the outcome of some equilibrium of the PAB auction if and only if it is the outcome of an equilibrium of the form given in the lemma. Therefore, we will restrict attention to equilibria b^* of the form for all $i \in N$:

$$b_i^*(y) = \begin{cases} p_i & \text{if } y = x_i \\ \sum_{j\in N} p_j & \text{if } y = K \\ 0 & \text{if } y \notin \{x_i, K\} \end{cases}$$

By construction of these bids, $w(b^*) = w(b^*_{-i})$ for all $i \in N$, which satisfies (II′). Therefore, b^* will be an equilibrium of the PAB auction that chooses assignment x and corresponding payments p with positive probability if and only if $x \in X(K)$, $p_i \geq 0$ for all $i \in N$, and $x \in \hat{X}(v_i, b^*_{-i})$ for all $i \in N$. The first two conditions guarantee that assignment x and payments p are feasible choices. The third condition ensures both that our tie-breaking rule picks x with positive probability and that b^* satisfies condition (I) so that it forms an equilibrium by Theorem 3(a).

It remains to prove that $x \in \hat{X}(v_i, b^*_{-i})$ for all $i \in N$ if and only if $p_i \leq v_i(x_i)$ for all $i \in N$ and (3) holds. Given the structure of b^*_{-i} and the fact that values satisfy free disposal, bidder i can win only subsets of $\{x_0, x_1, \ldots, x_n\}$ or nothing. Given bids (v_i, b^*_{-i}), bidder i would win x_i instead of nothing if and only if $v_i(x_i) \geq p_i$. Bidder i would win x_i instead of some subset of bundles $x_i \cup x_0 \cup (\cup_{j \in S} x_j)$ for some $S \subseteq N \setminus \{i\}$ if and only if $v_i(x_i) + \sum_{j \in S} p_j \geq v_i(x_i \cup x_0 \cup (\cup_{j \in S} x_j))$, which is equivalent to (3) if we require it for all $i \in N$ and all $S \subseteq N \setminus \{i\}$.

■

According to condition (3), successful bidders in the PAB auction have to pay at least any opponent's willingness to pay for their assigned bundles. In particular, the sum of payments in any equilibrium assignment must exceed the values of losing bidders for any union of assigned bundles. Payments must also be large enough to deter other winning bidders from wanting additional items. An equilibrium exists to support any individually rational outcome in which (i) all valuable items are assigned to some bidder and (ii) each group of bidders pays at least the incremental value that any other, individual bidder has for the items won by the group.

Not every assignment is possible in an equilibrium of the PAB auction because condition (3) must hold. For example, the PAB auction cannot, in equilibrium, assign all goods to bidders who have no value for them. As we will show in Theorem 7, BOCS and Vickrey auctions always have equilibria resulting in such a value-minimizing assignment. However, the condition from Theorem 6 does allow inefficient assignments. The condition considers only deviations in which a single bidder takes all bundles from some subset of the other bidders and does not take coordination among bidders' values into account. Two bidders might together be able to outbid a winning bidder, but as long as they both bid zero for what they want, neither bidder by himself has a profitable deviation.

Denote by $\Pi^A(v)$ the set of true payoff vectors in Nash equilibria of auction A for a given profile of values v. For the PAB auction, this set is:

$$\Pi^{PAB}(v) = \left\{ \left(\sum_{i \in N} p_i, v_1(x_1) - p_1, \ldots, v_n(x_n) - p_n \right) \mid (x, p) \text{ satisfies (1)–(3) of Theorem 6} \right\}$$

As a point of comparison to the set of PAB revenues, consider the Vickrey revenues when bidders follow their dominant strategies of bidding truthfully.

Corollary 2. *If $n > 2$, the equilibrium revenues from the PAB auction may be below the*

Vickrey auction revenues with truthful bidding:

$$\exists\ v \textit{ such that } \min\left\{\pi_0 \mid \pi = (\pi_0, \ldots, \pi_n) \in \Pi^{PAB}(v)\right\} < \pi_0^V(v).$$

Proof: The following is an example in which the revenue in the equilibrium b, shown below, of the PAB auction on is 1 but the Vickrey revenue with respect to the true values v is $\pi_0^V(v) = 2$.

	C	D	CD
v_1	2	2	4
v_2	1	0	1
v_3	0	1	1

	C	D	CD
b_1	0	0	1
b_2	0	0	1
b_3	0	0	1

■

In fact, even the revenues in efficient equilibria of the PAB auction may be lower than the Vickrey revenues with truthful bidding, as evidenced by the example in the proof of Corollary 2. For certain value profiles, every equilibrium of the PAB auction generates revenues that (weakly) exceed the Vickrey revenues with truthful bidding. For example, consider three bidders with values for bundles C, D, and CD of $v = ((2, 0, 2), (0, 2, 2), (0, 0, 3))$. The dominant-strategy equilibrium of the Vickrey auction leads to revenues of $\pi_0^V(v) = 2$. Any equilibrium of the PAB auction generates at least this revenue because at least one bidder does not win anything, thereby setting the lower bound on payments in condition (3) to 2. Any efficient equilibrium of the PAB auction must generate *strictly* more revenue, $\pi_0^{PAB} \geq 3$, because bidder 3 loses and has value 3 for bundle CD.

Corollary 2 implies that if $n > 2$, equilibrium payoffs from PAB auctions need not be in the core with respect to the true values, $\mathcal{C}(v)$. However, whenever the unique value-maximizing assignment gives all the items to one bidder, all equilibria have efficient outcomes and that bidder pays $p_i \in \left[p_i^V(v, x(v)), v_i(K)\right]$.[13] In this case, the payoffs are in the true core $\mathcal{C}(v)$. On the other hand, when $n = 2$, $\min\left\{\pi_0 \mid \pi = (\pi_0, \ldots, \pi_n) \in \Pi^{PAB}(v)\right\} \geq \pi_0^V(v)$, but the

[13] If K is assigned to bidder i, conditions (2) and (3) of Theorem 6 require $v_i(K) \geq p_i \geq \max_{j \in N \setminus \{i\}} v_j(K) = p_i^V(v, x(v))$. For any other assignment x, conditions (2) and (3) require $\sum_{j \in N \setminus \{i\}} v_j(x_j) \geq \sum_{j \in N \setminus \{i\}} p_j \geq v_i(K) - v_i(x)$, which is violated by the assumption that assigning K to bidder i is value maximizing: $v_i(K) > \max_{x \in X(K)} \sum_{i \in N} v_i(x_i)$.

equilibrium assignment need not be value maximizing.[14] This implies that the payoffs need not be in the true core.

BOCS auctions can admit even worse outcomes than the PAB auction in terms of efficiency and revenues. Thus, using the PAB auction instead of a BOCS auction eliminates some undesirable Nash equilibria without introducing any new ones. The following theorem illustrates this point through a type of folk theorem: any individually rational outcome that does not leave valuable items unassigned can occur in an equilibrium of a class of auctions that contains all BOCS auctions and the Vickrey auction.

Definition 6. *The class of auctions $\mathcal{B}$ consists of all sealed-bid auctions $(x(b), p(b, x(b)))$ that, for every $b \in B$, select $x(b) \in \hat{X}(b)$, break ties in favor of best responses, and choose payments $p_i(b, x(b)) \in \left[p_i^V(b, x(b)), p_i^A(b, x(b))\right]$ for some BOCS auction A and for all $i \in N$.*

The class of auctions $\mathcal{B}$ is a subset of class $\mathcal{A}$ that includes the Vickrey auction, all BOCS auctions, and all auctions whose payments lie between the two.

Theorem 7. *Given any auction in class $\mathcal{B}$, there exists a Nash equilibrium b^* that results in outcome $x = x(b^*)$ and $p = p(b^*, x(b^*))$ if and only if the following conditions hold:*

(1) $x \in X(K)$

(2) $p_i \in [0, v_i(x_i)] \quad \forall i \in N$

(3) $v_i(x_0 \cup x_i) = v_i(x_i) \quad \forall i \in N$

Proof: Note that $\mathcal{B} \subset \mathcal{A}$, so any equilibrium of an auction in $\mathcal{B}$ must satisfy (I) and (II) for all $i \in N$ by Theorem 2. Satisfying (I) for all $i \in N$ necessitates that the assignment x be feasible and that the unassigned items have no incremental value for any bidder, so (1) and (3) hold. Given that $x(b^*)$ fulfills condition (I), satisfying condition (II) for all $i \in N$ means $p_i(b^*, x(b^*)) = w(b^*_{-i}) - \sum_{j \neq i} b^*_j(x_j(b^*)) \in [0, v_i(x_i(b^*))]$ and condition (2) holds.

[14]When $n = 2$, condition (3) of Theorem 6 implies that $\pi_0 = p_1 + p_2 \geq [v_2(x_0 \cup x_1 \cup x_2) - v_2(x_2)] + [v_1(x_0 \cup x_1 \cup x_2) - v_1(x_1)] = w(v_1) + w(v_2) - [v_1(x_1) + v_2(x_2)] \geq w(v_1) + w(v_2) - w(v_1, v_2) = \pi_0^V(v)$. For an example of an equilibrium with an inefficient assignment, consider two bidders with values for bundles C, D, and CD of $v = ((1, 0, 1), (0, 1, 1))$ and bids $b = ((0, 0, 1), (0, 0, 1))$.

To show that (1)–(3) imply the existence of such an equilibrium for every auction $A \in \mathcal{B}$, it is sufficient to show that (1)–(3) imply the existence of such an equilibrium for every BOCS auction. Theorem 5 proves that, in any equilibrium of a BOCS auction, $p^{BOCS}(b^*, x(b^*)) = p^V(b^*, x(b^*))$ for all $x(b^*)$ chosen with positive probability. Thus, by definition, every $A \in \mathcal{B}$ chooses the assignment in the same way and selects $p^A(b^*, x(b^*)) = p^V(b^*, x(b^*))$ for all $x(b^*)$ chosen with positive probability, implying that b^* satisfies (I) and (II) for all $i \in N$.

Consider any feasible assignment $x \in X(K)$ such that $v_i(x_0 \cup x_i) = v_i(x_i)\ \forall i \in N$. Choose some $M > w(v)$. Let ω be the number of winning bidders in x. Relabel the winning bidders $1, \ldots, \omega$ and let $W = \{1, \ldots, \omega\}$. Then, for all $p_i \in [0, v_i(x_i)]$, the following is a Nash equilibrium of every BOCS auction with $x(b^*) = x$ and $p(b^*, x(b^*)) = p$:

$$\begin{aligned} b_i^*(x_i) &= M && \forall i \in \{1 \ldots, \omega\} \\ b_i^*(x_i \cup x_{i+1}) &= M + p_{i+1} && \forall i \in \{1, \ldots, \omega - 1\} \\ b_\omega^*(x_\omega \cup x_1) &= M + p_1 && \\ b_i^*(y) &= 0 && \forall i \in N \text{ and all other packages } y \end{aligned}$$

With these bids, the unique optimal assignment is $x(b^*) = x$. Since $M > w(v) \geq v_i(y)$ for all $y \in 2^K$ and all $i \in N$, $w(v_i, b_{-i}^*) = v_i(x_i \cup x_0) + (\omega - 1)M = v_i(x_i) + (\omega - 1)M$ for all $i \in W$ and $w(v_i, b_{-i}^*) = v_i(x_0) + \omega M = \omega M$ for all $i \notin W$. Therefore, $x \in \hat{X}(v_i, b_{-i}^*)$ for all $i \in N$ and b^* satisfies condition (I).

To show that condition (II) also holds, we prove, equivalently, that Vickrey payments $p_i^V(b^*, x_i) = w(b_{-i}^*) - \sum_{j \neq i} b_j^*(x_j)$ satisfy the core constraints (in which case, they will be the payments chosen by every BOCS auction and, therefore, by every auction in $\mathcal{B}$). Note that $w(b^*) = \omega M$, $\sum_{i \in N \setminus S} b_i^*(x_i) = (\omega - |S \cap W|)M$, and $w(b_{-S}^*) = (\omega - |S \cap W|)M + \sum_{i \in S} p_i$ for all $S \subset N$. Thus, $p_i^V(b^*, x_i) = (\omega - |\{i\} \cap W|)M + p_i - (\omega - |\{i\} \cap W|)M = p_i$. The core constraints are satisfied because $\sum_{i \in S} p_i^V(b^*, x_i) = \sum_{i \in S} p_i = (\omega - |S \cap W|)M + \sum_{i \in S} p_i - (\omega - |S \cap W|)M = w(b_{-S}^*) - \sum_{j \in N \setminus S} b_j^*(x_j)$ for all $S \subset N$. Thus, condition (II) holds for all $i \in N$. ■

These results about payoffs apply equally to BOCS auctions, the Vickrey auction, and all auctions in between. Thus, even though the Vickrey auction has a larger number of equilibria by Theorem 4, its extra equilibria do not add any outcomes to those achieved in the equilibria of the BOCS auctions.

The construction of the equilibrium in this proof uses bids that may seem implausibly large because $M > w(v)$. These large bids provide a convenient example that works for all assignments. However, many assignments can be implemented with more reasonable bids. For example, the efficient assignment and any individually rational payments will result from an equilibrium with the same structure as above, except replacing $b_i^*(x_i) = M$ with $b_i^*(x_i) = v_i(x_i)$.

The conditions in Theorem 7 do not impose any restrictions on payoff profiles. Any combination of payoffs can occur in equilibrium, even those that distribute the gains from trade in an extremely unequal way.

Corollary 3. *For all* $A \in \mathcal{B}$ *and all* v, $\Pi^A(v) = \left\{\pi \mid \pi_{-0} \in [0, v_1(x_1)] \times [0, v_2(x_2)] \times \ldots \times [0, v_n(x_n)]\,,\ \pi_0 = \sum_{i=1}^{n} (v_i(x_i) - \pi_i)\,,\ x \in X(K),\text{ and } v_i(x_0 \cup x_i) = v_i(x_i)\ \forall i \in N\right\}$. *This implies* $\left\{\pi_0 \mid \pi \in \Pi^A(v)\right\} = [0, w(v)]$ *and* $\left\{\pi_i \mid \pi \in \Pi^A(v)\right\} = [0, v_i(K)]$ *for all* $i \in N$.

Fixing an assignment, there are infinitely many equilibria that vary in payments from the unique bidder Pareto-best equilibrium, which has payoffs $\pi = (0, v_1(x_1), \ldots, v_n(x_n))$, to the unique bidder Pareto-worst equilibrium, which has payoffs $\pi = \left(\sum_{i\in N} v_i(x_i), 0, \ldots, 0\right)$. Anything in between these extremes is possible precisely because the bidders in our model are indifferent towards others' payoffs. Given the others' bids, a bidder can maximize his own payoff with infinitely many different bids, some of which give favorable payoffs to the other bidders and others of which raise their payments. For example, suppose some bidder places bids for all bundles in the auction that are higher than any other bidder's value for those bundles. In this extreme case, the other bidders do not want to bid high enough to win, so their bids serve only to determine the winning bidder's payment. They are indifferent between bidding zero, which gives the winner his highest payoff, and bidding the winner's true value for those bundles, which leaves him with a payoff of zero.

While the upper bound on revenues from auctions in $\mathcal{B}$, including the Vickrey and BOCS auctions, equals that of the PAB auction, the lowest equilibrium revenue from the PAB auction generally exceeds that from class $\mathcal{B}$ auctions. This occurs because auctions in class $\mathcal{B}$ allow more bid profiles to satisfy (I) and (II) and they choose lower revenues than the PAB auction for any given bids. Uncompetitive equilibria in which losing bidders fail to place bids, resulting in zero revenue for the seller, do not generally exist in the PAB auction, but always exist in auctions in $\mathcal{B}$. The payoffs from truthful bidding in the Vickrey auction also form an equilibrium outcome of all class $\mathcal{B}$ auctions, which does not always hold for the PAB auction.[15]

Corollary 4. *The payoff vector $\pi^V(v)$ resulting from truthful bidding by all $i \in N$ in the Vickrey auction is in $\Pi^A(v)$ for every auction $A \in \mathcal{B}$.*

Proof: Note that $x(v) \in X(K)$ is feasible, as is $p_i = p_i^V(v, x(v))$ because $p_i^V(v, x(v)) \leq v_i(x_i(v))$ for all $i \in N$. In auctions in class $\mathcal{B}$, the equilibrium payoffs when $x = x(v)$ and $p_i = p_i^V(v, x(v))$ equal those from truthful bidding in the Vickrey auction. ■

Thus, the payoffs we might expect from the Vickrey auction – those that result from its unique equilibrium in undominated strategies – are implemented by every auction in $\mathcal{B}$. As members of class $\mathcal{B}$, BOCS auctions fail to eliminate the poor auction outcomes that plague the Vickrey auction. Over all their equilibria, BOCS auctions implement exactly the same set of payoffs as the Vickrey auction. The final theorem summarizes this and other relationships between potential equilibrium payoffs.

Theorem 8.

(a) $\Pi^{PAB}(v) \subseteq \Pi^A(v) \subseteq \Pi^{A'}(v)$ *for all v, all $A \in \mathcal{A}$, and all $A' \in \mathcal{B}$.*

(b) *For every $A' \in \mathcal{B}$, there exists v such that $\Pi^{PAB}(v) \subset \Pi^{A'}(v)$.*

Proof: **(a)** By Theorem 4(a), $\text{NE}^{PAB}(v) \subseteq \text{NE}^A(v) \subseteq \text{NE}^V(v)$ for all v and all $A \in \mathcal{A}$. Combining this with Theorem 5, we find that any equilibrium of the PAB auction is an equilibrium with the same outcome of every auction in $\mathcal{A}$, and any equilibrium of any auction in $\mathcal{A}$ is an equilibrium with the same outcome of the Vickrey auction. The Vickrey auction

[15] See the example after Corollary 2 with $v = ((2,0,2),(0,2,2),(0,0,3))$.

belongs to class $\mathcal{B}$ and, by Theorem 7, any auction in class $\mathcal{B}$ has the same set of equilibrium outcomes. Therefore, $\Pi^{PAB}(v) \subseteq \Pi^{A}(v) \subseteq \Pi^{A'}(v)$.

(b) The following example shows that there exists v such that $\Pi^{PAB}(v) \subset \Pi^{V}(v)$. By Theorem 7, every auction in class $\mathcal{B}$, including the Vickrey auction, has the same set of equilibrium outcomes. Thus, the example implies there exists v such that $\Pi^{PAB}(v) \subset \Pi^{A'}(v)$ for all $A' \in \mathcal{B}$.

	C	D	CD
v_1	1	0	1
v_2	0	1	1

	C	D	CD
b_1	0	1	1
b_2	1	0	1

The bids above form a Nash equilibrium of the Vickrey auction in which both bidders win the good they do not value: $x_1(b) = B$ and $x_2(b) = A$. Payments and payoffs are zero. Achieving zero payoffs for the seller and both bidders requires zero gains from trade: $(x_0, x_1, x_2) \in \{(CD, \emptyset, \emptyset), (C, D, \emptyset), (D, \emptyset, C), (\emptyset, D, C)\}$. In the PAB auction, all four of these assignments require positive payments by condition (3) of Theorem 6 and, hence, negative payoffs. Therefore, it is not possible to achieve payoffs $(\pi_0, \pi_1, \pi_2) = (0, 0, 0)$ in any equilibrium of the PAB auction. ■

Any equilibrium of the PAB auction is an equilibrium with the same outcome of every auction in $\mathcal{A}$ (see Corollary 1 and Theorem 5). Since the PAB auction belongs to class $\mathcal{A}$, this implies the following corollary.

Corollary 5. *All auctions in $\mathcal{A}$ have Nash equilibrium outcomes in common: $(x(b^*), p(b^*, x(b^*)))$ is an equilibrium outcome of every auction in $\mathcal{A}$ if and only if it is an equilibrium outcome of the PAB auction.*

5. Extensions Regarding the Seller's Incentives and Budget Constraints

This section discusses the implications of our previous analyses for the seller's incentives to potentially manipulate the auction outcome after having collected the bids and for cases in which the bidders have binding budget constraints. The insights follow directly from our previous results.

5.1. The Seller's Commitment to the Auction Rules

Knowing the bids in the PAB auction, the seller maximizes his revenue by choosing the optimal assignment. In other auctions, such as the Vickrey auction and BOCS auctions, the seller might find it profitable to manipulate the outcome by disqualifying bidders (e.g., Ausubel and Milgrom, 2002), asking bidders to reduce their bids (e.g., Beck and Ott, 2009; Lamy, 2010), withholding items (e.g., Nozomu and Shirata, 2013), adding shill bids to increase the winners' payments, or switching tie-breaking rules. (The latter applies only to auctions in which revenues can vary with the way ties are broken. BOCS auctions that minimize the seller's revenue subject to the core constraints, the PAB auction, and the Vickrey auction do not fall into this category.[16]) However, our results prove that there exist equilibria that circumvent any such incentives to manipulate auctions in $\mathcal{A}$.

Theorem 9. *For every auction in $\mathcal{A}$, there exists a Nash equilibrium such that the seller, after collecting the bids, has no incentive to disqualify bidders or bids, to ask bidders to reduce their bids, to hold items back from sale, to add shill bids, or to alter the tie-breaking rule.*

Proof: Given any profile of bids b, $w(b)$ is the maximum revenue/payoff the seller can earn in every auction in $\mathcal{A}$. Disqualifying bidders or bids, reducing bids, and holding items back from sale can only decrease $w(b)$. Adding shill bids changes b to b' with $w(b') \geq w(b)$, but the artificial bids do not increase the maximum revenue, $w(b)$, that the seller may extract from the bidders. Altering the tie-breaking rule does not influence $w(b)$. Thus, such manipulations

[16] In all of these auctions, $\sum_{i\in N} p_i(b,x) = \sum_{i\in N} p_i(b,x')$ for all x, $x' \in \hat{X}(b)$. In the PAB auction, the revenue associated with all tied, optimal assignments is $\pi_0(b) = w(b)$. In the Vickrey auction, the revenue associated with all tied, optimal assignments is $\pi_0(b) = \sum_{i\in N}\left[w(b_{-i}) - \sum_{j\neq i} b_j(x_j(b))\right] = \sum_{i\in N} w(b_{-i}) - (n-1)w(b)$. In BOCS auctions, the solution to the problem of minimizing the seller's revenue subject to the core constraints, as given in (1), does not depend on the choice made among optimal assignments $x \in \hat{X}(b)$. BOCS auctions that minimize the seller's revenue subject to the core constraints are not the only BOCS auctions whose revenues do not vary with the way ties are broken. This property holds for all BOCS auctions in equilibrium (see Theorem 12) and for any BOCS auction such that the bidders' reported payoffs, $(\pi_1^r(b), \ldots, \pi_n^r(b))$, are the same for all $x \in \hat{X}(b)$.

can never help the seller when he earns $w(b)$, which he does in every equilibrium of the PAB auction (see also Bernheim and Whinston, 1986). By Theorem 8(a), the seller also receives a payoff of $w(b)$ in some equilibrium of every auction in $\mathcal{A}$. ■

5.2. Budget Constraints

Assume that each bidder has a fixed and finite budget $c_i > 0$. This budget impacts his payoff as follows. Bidder i's payoff is zero if he does not win any of the items, his payoff is $v_i(y) - p_i$ if he wins bundle y and pays price $p_i \leq c_i$, and his payoff is $-\infty$ if he wins a bundle y at a price $p_i > c_i$.

When budgets do not allow bidders to pay their full value for some of the bundles at auction, it may impact their bidding behavior and, thereby, the efficiency of the auction.[17] For example, truthful bidding is no longer a weakly dominant strategy in the Vickrey auction for bidders with a binding budget constraint, $v_i(y) > c_i$, for some y. However, every auction in class $\mathcal{B}$, including the Vickrey and BOCS auctions, has Nash equilibria that avoid this problem.

Corollary 6. *Every auction in class $\mathcal{B}$ has efficient Nash equilibria in which positive budget constraints are not binding: $p_i(b^*, x(b^*)) < c_i$ for all $i \in N$ and all $c_i > 0$.*

This corollary follows directly from Theorem 7, which implies that these auctions have Nash equilibrium outcomes with payments strictly below c_i for all $i \in N$ and all $c_i > 0$. This holds for every v and every possible equilibrium assignment, including the efficient assignment. Therefore, these auctions can achieve efficiency, even in the face of the strictest budget constraints.

[17] In BOCS auctions with incomplete information, even budgets that allow bidders to pay their full value for all possible bundles may impact their bidding behavior. For example, bidders with such budget constraints would not want to employ the equilibrium strategies with overbidding in the independent private values setting in Beck and Ott (2013).

6. Conclusion

This paper shows that all class $\mathcal{A}$ auctions – those that choose an optimal assignment given bids and have payments bounded above by bids and below by the minimum bid necessary to win the assigned bundle – share equilibria. Moreover, even the equilibria that are not shared among them have similar properties: each bidder wins what he would have won had he bid truthfully and pays his Vickrey payment, given the others' bids. Differences in incentives across the class $\mathcal{A}$ auctions stem from the different requirements on bids needed to achieve Vickrey payments, which result from variations in the auctions' payment rules.

The set of outcomes that result from the equilibria of the Vickrey auction and all BOCS auctions is the same and includes almost everything. Every feasible assignment that does not leave valuable goods unassigned, including giving all items to bidders who do not value them, is possible and can be accompanied by any payments that make the payoffs individually rational. This means the seller can get zero revenue or capture all of the surplus in equilibrium. The equilibrium outcomes of the PAB auction must satisfy a stronger condition, but still include many inefficient assignments and can involve payments that are outside of the core and even lower than the Vickrey payments with truthful bidding. The worst-case revenue in the PAB auction is at least as high as in any other class $\mathcal{A}$ auction.

The form of the equilibria of all class $\mathcal{A}$ auctions implies a very strong conclusion: any combinatorial auction that bounds payments below by the Vickrey payments should look just like a Vickrey auction in equilibrium. Therefore, information about bids alone can allow one to rule out a full-information Nash equilibrium. If any bidder's payment does not equal the price he would pay in a Vickrey auction with the same set of bids, then the bids do not form a full-information equilibrium. On the other hand, to affirmatively confirm that a given set of bids forms a Nash equilibrium, one needs information about the bidders' true values to verify condition (I).

Class $\mathcal{A}$ auctions have many types of equilibria. The most sensible strategies or most focal equilibria depend on the setting. Previous literature has focused on equilibria in which bidders place bids below their true values in order to cap their maximum possible payments. Using a truncation strategy in which bids for all bundles are reduced by the same amount prevents low bids from distorting the assignment (i.e., preserves condition (I)). Other equilibria with low bids can be extremely inefficient. Submitting a positive bid only for the bundle of all items is a bidder's best response if all other bidders do the same. Equilibria of this type effectively reduce the auction to a single unit (the bundle of all items).[18] Placing bids above true values can also be a sensible strategy and is part of many equilibria. In non-PAB auctions, aggressive bids may deter other bidders from competing for certain items, ensuring a win at a potentially very low price. In the Vickrey auction, bidders who want non-overlapping packages can submit high bids that reduce each other's payments.[19] More generally, across all class $\mathcal{A}$ auctions, bidders can have a large impact on their opponents' payments without altering the assignment or their own price. Coordination on iterated play of low-payment equilibria is likely in repeated settings. This holds even for the Vickrey auction: truthful bidding is no longer a dominant strategy of the repeated game. In other settings, bidders might have interactions outside the auction game, such as competing to sell similar products, that would incentivize them to try and raise their rivals costs.

The wide range of Nash equilibria and associated outcomes (assignments, payments, and payoffs) suggests that conclusions based on particular Nash equilibria or equilibrium refinements should be interpreted with caution. Common refinements differ across class $\mathcal{A}$ auctions: analyses of the Vickrey auction usually focus on equilibria in undominated strategies whereas analyses of the PAB and BOCS auctions typically select equilibria in truncation strategies. These refinements select different equilibria,[20] so it might be fruitful to instead find a single refinement that can be applied to all class $\mathcal{A}$ auctions. This paper provides the basis for such analyses by describing the complete set of full-information equilibria.

[18] PAB, BOCS, and Vickrey auctions all have Bayesian equilibria of this type for any number of bidders or items. See Bernheim and Whinston (1986), Beck and Ott (2013), Holzman and Monderer (2004), respectively.

[19] For example, suppose there are three bidders, 1, 2, and 3, and two items, C and D, and bids are written $b_i = (b_{iC}, b_{iD}, b_{iCD})$. By bidding $b_1 = (x, 0, x)$, $b_2 = (0, x, x)$, and $b_3 = (0, 0, y)$, bidders 1 and 2 place complementary bids (i.e., for non-overlapping packages), which causes them to win and pay zero whenever $y \leq x$.

[20] For example, Beck and Ott (2013) find equilibria in undominated strategies of BOCS auctions that are not truncation strategies but involve bidding more than a bundle's value. In PAB auctions, bid shading by the same amount for all bundles as in truncation equilibria is not necessary for equilibria in undominated strategies.

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Appendix

A. Properties and Implementation of the Tie-Breaking Rule

Equilibria of combinatorial auctions usually depend on the tie-breaking rule. In this paper, we break ties in favor of best responses to maximize the set of Nash equilibria. However, the majority of our results do not depend on this tie-breaking rule. Our characterizations of best responses and equilibria hold for all tie-breaking rules because the proofs of Theorems 1, 2, 3(a) and 3(b) do not rely on the way the auction breaks ties. Theorems 3(c), 4, 5, and 8 require that the auctions being compared resolve ties in the same way. On the other hand, certain tie-breaking rules destroy the existence of equilibria in Theorems 2 and 9. The conditions on equilibrium outcomes in Theorems 6 and 7 are always necessary, but sufficiency depends on the tie-breaking rule.

In this appendix, we prove our claim that breaking ties in favor of best responses maximizes the set of equilibria (Theorem 10(a)). Thus, using an alternate tie-breaking rule will never add equilibria, but it might eliminate some of our equilibria. We identify those alternate tie-breaking rules that do not destroy equilibria (Theorem 10(b)): those that choose an assignment that fulfills conditions (I) and (II) for all $i \in N$ whenever possible. A popular rule in equilibrium analysis is tie breaking in favor of efficiency, which chooses the tied assignment that generates the highest (true) value to the bidders. Though any assignment chosen in equilibrium must maximize the sum of true values among tied assignments to prevent profitable deviations (Theorem 11),[21] this does not imply that every value maximizing assignment among $\hat{X}(b^*)$ supports an equilibrium. However, in certain auctions, breaking ties in favor of efficiency results in the same set of equilibria as breaking ties in favor of best responses (Theorem 12). Finally, we show that breaking ties in favor of best responses can be implemented by an extended game in which bidders name, in addition to their bids, their desired bundle in case of a tie (Theorem 13).

For these theorems, we define an extended class of auctions, $\tilde{\mathcal{A}}$, that allows for any tie-breaking rule. Since class $\mathcal{A}$ contains only auctions that break ties in favor of best responses, $\mathcal{A} \subset \tilde{\mathcal{A}}$.

Definition 7. *The class of auctions $\tilde{\mathcal{A}}$ consists of all sealed-bid auctions $(x(b), p(b, x(b)))$ that, for every $b \in B$, select $x(b) \in \hat{X}(b)$ and choose payments $p_i(b, x(b)) \in \left[p_i^V(b, x(b)), b_i(x_i(b))\right]$*

[21] Bernheim and Whinston (1986, Footnote 5, p. 5) discussed this property of Nash equilibria in relation to PAB auctions.

for all $i \in N$.

Define the (potentially empty) set of tied assignments that fulfill $x \in \hat{X}(v_i, b_{-i})$ and $p_i(b,x) = p_i^V(b,x)$ for all $i \in N$ as follows:

$$\hat{X}^*(b,v) := \left\{ x \mid x \in \hat{X}(b) \wedge x \in \hat{X}(v_i, b_{-i}) \ \forall i \in N \wedge p_i(b,x) = p_i^V(b,x) \ \forall i \in N \right\}.$$

Note that $\hat{X}^*(b,v)$ is the set of assignments over which tie breaking in favor of best responses randomizes, assuming $\hat{X}^*(b,v) \neq \emptyset$. It is the set of assignments such that conditions (I) and (II) are satisfied for all $i \in N$. Therefore, by the proof of Theorem 2, for any $\tilde{A} \in \tilde{\mathcal{A}}$,

(NE) $\quad b \in \text{NE}^{\tilde{A}}$ if and only if $x(b) \in \hat{X}^*(b,v)$ for all $x(b)$ chosen with positive probability.

The next theorem shows the relationship between the Nash equilibria of auctions that differ only in the way they break ties.

Theorem 10. *Consider any* $\tilde{A} \in \tilde{\mathcal{A}}$ *and* $A \in \mathcal{A}$ *such that* $p^A(b,x) = p^{\tilde{A}}(b,x)$.

(a) $NE^{\tilde{A}}(v) \subseteq NE^A(v)$ *for all* v.

(b) $NE^{\tilde{A}}(v) = NE^A(v)$ *if and only if* $x(b) \in \hat{X}^*(b,v)$ *for all* $b \in NE^A(v)$ *and all* $x(b)$ *chosen by* $\tilde{A}$ *with positive probability.*

Proof: (a) Consider any $b \in \text{NE}^{\tilde{A}}$. By (NE), $\tilde{A}$ selects $x(b)$ from $\hat{X}^*(b,v)$, so $\hat{X}^*(b,v) \neq \emptyset$ and $b \in \text{NE}^A$. (b) The result follows directly from part (a) and (NE). ■

The following intuition underlies Theorem 10(a). Tie breaking in favor of best responses chooses (randomly) among all tied assignments in $\hat{X}^*(b,v)$, if such assignments exist. Thus, in any equilibrium, it places positive probability on *every* assignment that leaves bidders with no incentives to deviate. If any alternate tie-breaking rule assigns positive probability to assignments that are given probability zero by tie breaking in favor of best responses, then at least one bidder has a profitable deviation and equilibrium is destroyed.

Theorem 10(b) shows that, given v, all tie-breaking rules that choose an assignment from $\hat{X}^*(b,v)$ whenever such an assignment exists will lead to the same equilibria. The way in which they choose among assignments in $\hat{X}^*(b,v)$ and the assignment selected if $\hat{X}^*(b,v)=\emptyset$ do not matter. Other tie-breaking rules will eliminate some equilibria. For example, if the efficient (total-value maximizing) assignment is unique, then applying a tie-breaking rule that chooses the total-value minimizing assignment will destroy all efficient, pure-strategy equilibria of the PAB auction. In the Vickrey auction, truthful bidding is an efficient equilibrium independent of the tie-breaking rule. The same holds for BOCS auctions whenever the true core contains the true Vickrey payoff vector.

Denote by $\hat{X}^{\text{eff}}(b,v)=\arg\max_{x\in\hat{X}(b)}\sum_{i\in N}v_i(x_i)$ the set of value-maximizing assignments among those in $\hat{X}(b)$. The next theorem demonstrates that this set contains every equilibrium assignment of every auction in $\tilde{\mathcal{A}}$. This result stems from the close connection between condition (I) and value maximization, as shown in part (a) of the theorem.

Theorem 11.

(a) *If* $\hat{X}(b)\cap\left\{\cap_{i\in N}\hat{X}(v_i,b_{-i})\right\}\neq\emptyset$*, then* $\hat{X}(b)\cap\left\{\cap_{i\in N}\hat{X}(v_i,b_{-i})\right\}=\hat{X}^{\textit{eff}}(b,v)$.

(b) $\hat{X}^*(b,v)\subseteq\hat{X}^{\textit{eff}}(b,v)$ *for all* $\tilde{A}\in\tilde{\mathcal{A}}$

Proof: (a) We will first show that $\hat{X}(b)\cap\left\{\cap_{i\in N}\hat{X}(v_i,b_{-i})\right\}\subseteq\hat{X}^{\text{eff}}(b,v)$.[22] Consider any $x\in\hat{X}(b)\cap\left\{\cap_{i\in N}\hat{X}(v_i,b_{-i})\right\}$. Since $x\in\hat{X}(v_i,b_{-i})$ for all $i\in N$, we know that $v_i(x_i)+\sum_{j\neq i}b_j(x_j)\geq v_i(\hat{x}_i)+\sum_{j\neq i}b_j(\hat{x}_j)$ for all $i\in N$ and all $\hat{x}\in\hat{X}(b)$. Summing over i leads to $\sum_{i\in N}\left(v_i(x_i)+\sum_{j\neq i}b_j(x_j)\right)\geq\sum_{i\in N}\left(v_i(\hat{x}_i)+\sum_{j\neq i}b_j(\hat{x}_j)\right)$. Rearranging terms and using the fact that $x,\hat{x}\in\hat{X}(b)$, we get $\sum_{i\in N}\left(v_i(x_i)-v_i(\hat{x}_i)\right)\geq\sum_{i\in N}\left(\sum_{j\neq i}b_j(\hat{x}_j)-\sum_{j\neq i}b_j(x_j)\right)=(n-1)\sum_{i\in N}\left(b_i(\hat{x}_i)-b_i(x_i)\right)=0$. Thus, $x\in\hat{X}^{\text{eff}}(b,v)$.

Second, we will show that if $\hat{X}(b)\cap\left\{\cap_{i\in N}\hat{X}(v_i,b_{-i})\right\}\neq\emptyset$, then $\hat{X}(b)\cap\left\{\cap_{i\in N}\hat{X}(v_i,b_{-i})\right\}\supseteq\hat{X}^{\text{eff}}(b,v)$. By definition, $\sum_{i\in N}b_i(x_i)=\sum_{i\in N}b_i(x_i')$ and $\sum_{i\in N}v_i(x_i)=\sum_{i\in N}v_i(x_i')$ for

[22] $\hat{X}^{\text{eff}}(b,v)\neq\emptyset$ by definition, but $\hat{X}(b)\cap\left\{\cap_{i\in N}\hat{X}(v_i,b_{-i})\right\}$ may be empty. An example in which no $x\in\hat{X}(b)$ satisfies $x\in\hat{X}(v_i,b_{-i})$ for all $i\in N$ is the following:

	C	D	CD
v_1	2	1	2
v_2	2	1	2

	C	D	CD
b_1	0	1	1
b_2	1	2	2

$\hat{X}(b)=\{(\emptyset,C,D),(\emptyset,D,C),(\emptyset,\emptyset,CD),(C,\emptyset,D)\}$
$\hat{X}^{\text{eff}}(b,v)=\{(\emptyset,C,D),(\emptyset,D,C)\}$
$\hat{X}(v_1,b_2)=\{(\emptyset,C,D)\}$
$\hat{X}(b_1,v_2)=\{(\emptyset,D,C)\}$

all $x,\, x' \in \hat{X}^{\text{eff}}(b,v) \subseteq \hat{X}(b)$. Thus, $\sum_{i\in N} v_i(x_i) - (n-1)\sum_{i\in N} b_i(x_i) = \sum_{i\in N} v_i(x'_i) - (n-1)\sum_{i\in N} b_i(x'_i)$. Rearranging terms, we get:

$$\sum_{i\in N}\Big(v_i(x_i) + \sum_{j\neq i} b_j(x_j)\Big) = \sum_{i\in N}\Big(v_i(x'_i) + \sum_{j\neq i} b_j(x'_j)\Big) \quad \forall\, x,\, x' \in \hat{X}^{\text{eff}}(b,v). \tag{7}$$

The set $\hat{X}(b) \cap \Big\{\cap_{i\in N}\hat{X}(v_i, b_{-i})\Big\}$ is non-empty by assumption and contained in $\hat{X}^{\text{eff}}(b,v)$ by the proof above, so there exists $x \in \hat{X}^{\text{eff}}(b,v)$ such that $v_i(x_i) + \sum_{j\neq i} b_j(x_j) \geq v_i(x'_i) + \sum_{j\neq i} b_j(x'_j)$ for all $i \in N$ and all $x' \in X(K)$. Summing up weakly lower summands and getting the same sum in (7) requires that all summands be equal: $v_i(x_i) + \sum_{j\neq i} b_j(x_j) = v_i(x'_i) + \sum_{j\neq i} b_j(x'_j)$ for all $i \in N$. Thus, $x' \in \hat{X}^{\text{eff}}(b,v)$ implies $x' \in \hat{X}(v_i, b_{-i})$ for all $i \in N$.

(b) When $\hat{X}^*(b,v) = \emptyset$, the result is obvious.
When $\hat{X}^*(b,v) \neq \emptyset$, then $\hat{X}(b) \cap \Big\{\cap_{i\in N}\hat{X}(v_i, b_{-i})\Big\} \neq \emptyset$ and the result follows directly from (a) and the fact that $\hat{X}^*(b,v) = \hat{X}(b) \cap \Big\{\cap_{i\in N}\hat{X}(v_i, b_{-i})\Big\} \cap \big\{x \mid p_i(b,x) = p_i^V(b,x)\ \forall i \in N\big\}$. ■

A common way of resolving ties is to break them in favor of efficiency, as done, for example, by Bernheim and Whinston (1986). Denote by $A^{\text{eff}} \in \tilde{\mathcal{A}}$ any auction that breaks ties in favor of efficiency (i.e., that chooses randomly among all $x \in \hat{X}^{\text{eff}}(b,v)$). The set of equilibria generated by breaking ties in favor of efficiency will be the same as that generated by breaking ties in favor of best responses for any auction satisfying a particular condition on prices, as shown in the next theorem.

Theorem 12. *Consider any auction $A^{\text{eff}} \in \tilde{\mathcal{A}}$ and the auction $A \in \mathcal{A}$ such that $p^A(b,x) = p^{A^{\text{eff}}}(b,x)$. $NE^{A^{\text{eff}}}(v) = NE^A(v)$ if and only if $p_i(b,x) = p_i^V(b,x)$ for all $i \in N$, all $x \in \hat{X}^{\text{eff}}(b,v)$, and all b such that $\hat{X}^*(b,v) \neq \emptyset$. Auctions that have this property include the Vickrey, BOCS, and PAB auctions and any other auction in which $\sum_{i\in N} p_i(b,x) = \sum_{i\in N} p_i(b,\hat{x})$ for all $x,\, \hat{x} \in \hat{X}^{\text{eff}}(b,v)$ and for all b such that $\hat{X}^*(b,v) \neq \emptyset$.*

Proof: By Theorem 10(b), $\text{NE}^{A^{\text{eff}}}(v) = \text{NE}^A(v)$ if and only if $\hat{X}^{\text{eff}}(b,v) \subseteq \hat{X}^*(b,v)$ for all $b \in \text{NE}^A(v)$. Note that $\hat{X}^*(b,v) = \hat{X}(b) \cap \Big\{\cap_{i\in N}\hat{X}(v_i, b_{-i})\Big\} \cap \big\{x \,|\, p_i(b,x) = p_i^V(b,x)\ \forall i \in N\big\}$. For every $b \in \text{NE}^A(v)$, $\hat{X}^*(b,v) \neq \emptyset$ and this implies, by Theorem 11(a), that $\hat{X}^{\text{eff}}(b,v) = \hat{X}(b) \cap \Big\{\cap_{i\in N}\hat{X}(v_i, b_{-i})\Big\}$. Therefore, $\hat{X}^{\text{eff}}(b,v) \subseteq \hat{X}^*(b,v)$ for all $b \in \text{NE}^A(v)$ if and only if

$p_i(b,x) = p_i^V(b,x)$ for all $i \in N$, all $x \in \hat{X}^{\text{eff}}(b,v)$, and all $b \in \text{NE}^A(v)$.

Next, we show that whenever an auction's payment rule is such that $\sum_{i\in N} p_i(b,x) = \sum_{i\in N} p_i(b,\hat{x})$ for all x, $\hat{x} \in \hat{X}^{\text{eff}}(b,v)$ and for all b such that $\hat{X}^*(b,v) \neq \emptyset$, then $p_i(b,x) = p_i^V(b,x)$ for all $i \in N$, all $x \in \hat{X}^{\text{eff}}(b,v)$, and all b such that $\hat{X}^*(b,v) \neq \emptyset$. This follows from four facts. First, by assumption, total payments in the considered auction do not depend on the chosen assignment in $\hat{X}^{\text{eff}}(b,v)$. Second, by the assumption that $\hat{X}^*(b,v) \neq \emptyset$ and Theorem 11(b), there exists $x' \in \hat{X}^*(b,v) \subseteq \hat{X}^{\text{eff}}(b,v)$ such that $p_i(b,x') = p_i^V(b,x')$ for all $i \in N$, which implies $\sum_{i\in N} p_i(b,x') = \sum_{i\in N} p_i^V(b,x')$. Third, the total Vickrey revenue $\pi_0^V(b) = \sum_{i\in N} p_i^V(b,x) = \sum_{i\in N}\left[w(b_{-i}) - \sum_{j\neq i} b_j(x_j(b))\right] = \sum_{i\in N} w(b_{-i}) - (n-1)w(b)$ does not depend on the assignment $x \in \hat{X}(b)$, so $\sum_{i\in N} p_i^V(b,x) = \sum_{i\in N} p_i^V(b,\hat{x})$ for all x, $\hat{x} \in \hat{X}^{\text{eff}}(b,v) \subseteq \hat{X}(b)$. Fourth, since $A \in \mathcal{A}$, payments must satisfy $p_i(b,x) \geq p_i^V(b,x)$. Thus, it follows from these four facts that $\sum_{i\in N} p_i(b,x) = \sum_{i\in N} p_i^V(b,x)$ and $p_i(b,x) \geq p_i^V(b,x)$ for all $i \in N$ and all $x \in \hat{X}^{\text{eff}}(b,v)$, which implies $p_i(b,x) = p_i^V(b,x)$ for all $i \in N$ and all $x \in \hat{X}^{\text{eff}}(b,v)$.

It remains to prove that in the Vickrey, BOCS, and PAB auctions, $\sum_{i\in N} p_i(b,x) = \sum_{i\in N} p_i(b,\hat{x})$ for all x, $\hat{x} \in \hat{X}^{\text{eff}}(b,v)$ and for all b such that $\hat{X}^*(b,v) \neq \emptyset$. In the Vickrey or PAB auction, the revenue resulting from any $x \in \hat{X}(b)$ is $\pi_0^V(b) = \sum_{i\in N} w(b_{-i}) - (n-1)w(b)$ or $\pi_0(b) = \sum_{i\in N} b_i(x_i(b)) = w(b)$, respectively, and does not vary with the assignment. Consider any BOCS auction. If $\hat{X}^*(b,v) \neq \emptyset$, then there exists $x \in \hat{X}(b)$ such that $p_i(b,x) = p_i^V(b,x)$ for all $i \in N$. Therefore, the bidder-optimal frontier of the core $\mathcal{C}(b)$ is the singleton $\pi^{r,V}(b)$ and $\pi_0^V(b) = \pi_0^{r,V}(b)$ does not depend on the chosen assignment $x \in \hat{X}(b)$. ■

By the proof of Theorem 12, $\text{NE}^{A^{\text{eff}}}(v) = \text{NE}^A(v)$ if and only if $\hat{X}^{\text{eff}}(b,v) = \hat{X}^*(b,v)$ for all b such that $\hat{X}^*(b,v) \neq \emptyset$. This is because $\hat{X}^{\text{eff}}(b,v) \supseteq \hat{X}^*(b,v)$ by Theorem 11, so the requirement that $\hat{X}^{\text{eff}}(b,v) \subseteq \hat{X}^*(b,v)$ will hold only if $\hat{X}^{\text{eff}}(b,v) = \hat{X}^*(b,v)$. Therefore, if the two tie-breaking rules result in the same set of equilibria, they randomize among exactly the same set of assignments in equilibrium. Out of equilibrium, these two tie-breaking rules may select different assignments.[23]

[23] An out-of-equilibrium example in which the two tie-breaking rules choose different assignments is the

The theorem shows that, in many popular auctions, breaking ties in favor of best responses is equivalent to breaking ties in favor of efficiency. However, there are core-selecting auctions for which this is not the case. Consider the core-selecting auction in which bidders pay their bids unless a single bidder wins all of the items, and in that case, the winning bidder pays its Vickrey payment. Suppose the underlying values and bids are as follows:

	C	D	CD
v_1	1	0	3
v_2	0	2	2

	C	D	CD
b_1	1	0	3
b_2	0	2	2

Since the bids are truthful, both optimal assignments are efficient. If the auction breaks ties in favor of efficiency, it will randomize between $(\emptyset, C, D)$ and $(\emptyset, CD, \emptyset)$. Truthful bidding is not an equilibrium because bidder 1 would prefer to reduce his bid for C to guarantee that he wins both items and pays his Vickrey payment instead of winning C and paying his full value for it. Breaking ties in favor of best responses rules out $(\emptyset, C, D)$ because of this profitable deviation and instead chooses $(\emptyset, CD, \emptyset)$ with probability 1, which allows truthful bidding to be a Nash equilibrium of the auction. This auction violates the condition on revenues in Theorem 12 because the seller earns $\sum_i b_i(x_i) = 1+2 = 3$ when the assignment is $x = (\emptyset, C, D)$ whereas he earns only $p_1^V(b, x) = 2$ when the assignment is $x = (\emptyset, CD, \emptyset)$.

Breaking ties in favor of either best responses or efficiency requires knowledge of the bidders' underlying values. It might seem unreasonable for an auctioneer that knows only the reported bids to implement such tie-breaking rules. The next theorem shows, however, that both of

following:

	C	D	CD
v_1	2	0	2
v_2	0	2	2
v_3	0	0	5

	C	D	CD
b_1	1	0	1
b_2	0	1	1
b_3	0	0	2

$\hat{X}(b) = \{(\emptyset, C, D, \emptyset), (\emptyset, \emptyset, \emptyset, CD)\}$
$\hat{X}(v_1, b_{-1}) = \{(\emptyset, C, D, \emptyset)\}$ $\quad \hat{X}^{\text{eff}}(b, v) = \{(\emptyset, \emptyset, \emptyset, CD)\}$
$\hat{X}(v_2, b_{-2}) = \{(\emptyset, C, D, \emptyset)\}$ $\quad \hat{X}^*(b, v) = \emptyset$
$\hat{X}(v_3, b_{-3}) = \{(\emptyset, \emptyset, \emptyset, CD)\}$

The payments associated with each of the optimal assignments must equal bids in every auction in $\tilde{\mathcal{A}}$ because these particular bids equal the Vickrey payments. Therefore, breaking ties in favor of best responses means selecting $(\emptyset, C, D, \emptyset)$ whereas breaking ties in favor of efficiency means choosing $(\emptyset, \emptyset, \emptyset, CD)$.

these rules – the former rule for all $A \in \mathcal{A}$ and the latter rule for all $A^{\text{eff}} \in \tilde{\mathcal{A}}$ that satisfy the conditions of Theorem 12 – can be implemented based on the information provided by bidders in an equilibrium of an extended game. The messages in this mechanism consist of bids and additional "nontype" messages.

Consider the following game A^{TBG} that is composed of an auction $A \in \mathcal{A}$ and a tie-breaking game. Each bidder i simultaneously reports his bids b_i: $2^K \to \mathbb{R}^{2^K}$ and a set of preferred tied assignments, $R_i\left(\hat{X}\left(b_i, b'_{-i}\right)\right) \subseteq \hat{X}\left(b_i, b'_{-i}\right)$, for every feasible profile of bids $\left(b_i, b'_{-i}\right)$. The extended games' tie-breaking rule then chooses randomly from the set $\cap_{i \in N} R_i\left(\hat{X}(b)\right) \subseteq \hat{X}(b)$ or from $\hat{X}(b)$ if the former set is empty.

We show that for any Nash equilibrium b^* of any auction $A \in \mathcal{A}$, the corresponding game A^{TBG} has a Nash equilibrium in which (i) bidders bid b^* and report their true preferences among tied assignments for every possible set of opposing bids, (ii) if b^* results in a tie, there is at least one tied assignment on which all bidders agree, and (iii) the set of assignments on which bidders agree equals $\hat{X}^*(b^*, v)$.

Theorem 13. *Consider any $A \in \mathcal{A}$. For any $b^* \in NE^A(v)$, (b^*, R^*) with $R_i^*\left(\hat{X}(b_i^*, b_{-i})\right) = \arg\max_{x \in \hat{X}(b_i^*, b_{-i})} v_i(x_i) - p_i((b_i^*, b_{-i}), x)$ for all b_{-i} and all i is a Nash equilibrium of A^{TBG} and results in the same payoffs as equilibrium b^* of A. If A is a Vickrey, BOCS, or PAB auction, then the seller receives the same revenue regardless of how ties are broken among $\hat{X}(b^*)$.*

Proof: Consider any $A \in \mathcal{A}$ and any $b^* \in \text{NE}^A(v)$. By (NE), A chooses an assignment from $\hat{X}^*(b^*, v)$ and $\hat{X}^*(b^*, v) \neq \emptyset$. Assume that all bidders i in the game A^{TBG} report b_i^* and $R_i^*\left(\hat{X}(b_i^*, b_{-i})\right)$ for all b_{-i}. For each i, $R_i^*\left(\hat{X}(b^*)\right)$ contains exactly the assignments x that fulfill $x \in \hat{X}(v_i, b_{-i}^*)$ and $p_i(b^*, x(b^*)) = p_i^V(b^*, x(b^*))$ by the proof of Theorem 1(a). Thus, $\cap_{i \in N} R_i^*\left(\hat{X}(b^*)\right) = \hat{X}^*(b^*, v)$, which implies A^{TBG} chooses $x(b^*) \in \hat{X}^*(b^*, v)$. It follows that each bidder i receives $\pi_i^V(v_i, b_{-i}^*)$ and cannot profitably deviate because $\pi_i^V(v_i, b_{-i}^*)$ is his maximum payoff, given b_{-i}^*.

By the proof of Theorem 12, for any equilibrium b^* of the Vickrey, BOCS, or PAB auctions, $\pi_0(b^*, x) = \pi_0(b^*, \hat{x})$ for all x, $\hat{x} \in \hat{X}(b^*)$, so the seller is indifferent between all $x \in \hat{X}(b^*)$. ■